FUZZY VECTOR MEDIAN DEFINITION BASED ON A FUZZY VECTOR DISTANCE

Vassilios Chatzis and Ioannis Pitas

Department of Informatics, University of Thessaloniki
540 06 Thessaloniki, GREECE
phone,fax: +30-31-996304, e-mail: [chatzis,pitas]@zeus.csd.auth.gr

ABSTRACT

In this paper, the Fuzzy Vector Median is proposed, defined as an extension of Vector Median. It is based on a novel distance definition of multidimensional fuzzy numbers (fuzzy vectors), which satisfy the property of angle decomposition. The proposed distance of two fuzzy vectors depends on the classical distance of the fuzzy set centers and on the fuzziness that every fuzzy set holds. As a result the Fuzzy Vector Median of a set of fuzzy vectors is affected by the presence of fuzziness.

1. INTRODUCTION

Multichannel signals appear in many important signal processing applications. Typical examples are the multispectral satellite images, color images and signals that represent velocity. Multichannel techniques, that have been proposed rather recently, and consider the correlation of the channels, seem to be the most appropriate way to process multichannel signals. One of the most popular technique is the vector median filter, that inherently utilizes the correlation of the channels and gives some desirable properties such as, the zero impulse response and the preservation of the signal edges [6].

However, any crisp value conceals a degree of uncertainty that can be described by using fuzzy numbers [2]-[5]. In this paper, the uncertainty of the vector value will be taken into account by using fuzzy instead of crisp vectors. The term *fuzzy vector* will be used in the following, to describe the extension of an n-dimensional crisp set $\mathbf{C}$ to an n-dimensional fuzzy set $\mathbf{X}$ defined in an $(n+1)$-dimensional hyperspace, by using a membership function $\mu : \mathbf{C} \rightarrow [0,1]$ [1]. The term fuzzy vector is usually found in the literature, describing the notion of a vector of n 1-dimensional fuzzy numbers. This notion could be appropriate to describe the uncertainty in non-correlated data or when different degrees of uncertainty is possible to be given to each signal channel.

2. ANGLE DECOMPOSED FUZZY VECTORS (ADFV)

2.1. Definition of ADFVs

The fuzzy sets are usually described by the union of their α-cuts instead of the membership function. The α-cuts of an 1-dimensional fuzzy set are the classical sets X^α, where $x \in X^\alpha \Leftrightarrow \mu(x) \geq \alpha$. They can easily extended to describe multidimensional fuzzy sets. Thus, the α-cuts of an n-dimensional fuzzy set will be the classical sets $\mathbf{X}^\alpha$, where $\mathbf{x} \in \mathbf{X}^\alpha \Leftrightarrow \mu(\mathbf{x}) \geq \alpha$, and μ is a function of n variables. A fuzzy set is called *normal* if $\exists \mathbf{x} : \mu(\mathbf{x}) = 1$ or $\mathbf{X}^1 \neq \emptyset$. It is called *convex* if $\forall \alpha_1, \alpha_2 \in [0,1], \alpha_1 > \alpha_2 \Leftrightarrow \mathbf{X}^{\alpha_1} \subseteq \mathbf{X}^{\alpha_2}$. A normal and convex fuzzy set is called *fuzzy number* [7]-[9]. A 1-dimensional fuzzy number will be called *convex fuzzy number* when the corresponding α-cuts are convex sets. In the following the Angle Decomposed Fuzzy Vectors (ADFVs) will be defined as a subset of multidimensional fuzzy numbers and will provide us the ability to define a distance between them.

Let $\mathbf{X}$ be an n-dimensional fuzzy set, $\mu_{\mathbf{X}}(\mathbf{x})$ its membership function and $\mathbf{X}^\alpha$ the corresponding α-cuts. Consider also that there is only one vector $\mathbf{x}_c$ where $\mu_{\mathbf{X}}(\mathbf{x}_c) = 1$. The vector $\mathbf{x}_c$ will be called the center of the fuzzy set. Consider also $n-1$ angles $\boldsymbol{\theta} = (\theta_i, i = 1, 2, \ldots, (n-1))$, $\theta_i \in [0, \pi)$. The centre of the fuzzy set $\mathbf{x}_c$ and each angle θ_i determine a hyperplane. The union of $n-1$ hyperplanes is a straight line (direction) in the n-dimensional hyperspace, where a function μ_1 can be defined as $\mu_1(x, \theta) = \mu_X(x_1(x,\theta), x_2(x,\theta), \ldots,\ x_{n-1}(x,\theta),\ x)$. This function can be considered as a membership function of an 1-dimensional fuzzy set X^θ. Then, the ADFVs are defined as follows:

Definition 1: An n-dimensional fuzzy set $\mathbf{X}$ is an Angle Decomposed Fuzzy Vector (ADFV), if, for each vector of angles $\theta = (\theta_1, \theta_2, \ldots, \theta_{n-1})$, the 1-d fuzzy set $X^\theta = \{x, \mu_1(x, \theta)\}$ is a convex fuzzy number.

An example of a 3-dimensional ADFV and the angle decomposed 1-d convex fuzzy numbers is shown in Figure 1. It is easy to prove that any ADFV is a fuzzy vector. We

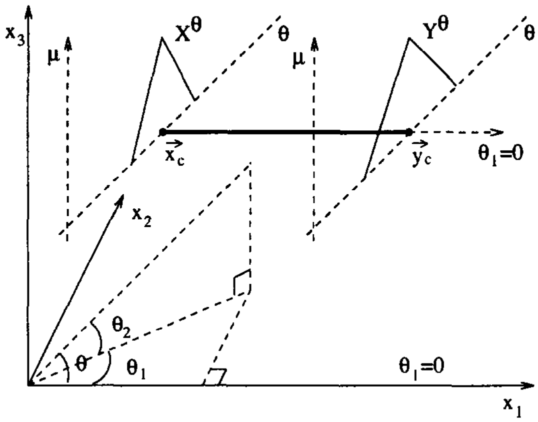


Figure 1: Two 1-d convex fuzzy numbers X^{θ},Y^{θ}, coming from two 3-d ADFVs $\mathbf{X}$,$\mathbf{Y}$ when the angle vector $\theta = (\theta_1, \theta_2)$ is determined.

can also prove that if θ is a $n-1-k$ vector the function $\mu_k(x_1, x_2, \ldots, x_k, \theta)$ can be considered as a membership function of a k-dimensional ADFV. The use of ADFVs give us the ability to establish a one to one correspondence between the points of two ADFVs that limit their α-cuts on a certain direction. By using this correspondence, a distance measure between multidimensional ADFVs will be defined.

2.2. ADFVs distance definition and properties

Let us assume that X, Y are 1-d fuzzy numbers, symbolized as $X = \bigcup_{\alpha} \cdot [x_l^{\alpha}, x_r^{\alpha}]$, $Y = \bigcup_{\alpha} \cdot [y_l^{\alpha}, y_r^{\alpha}]$, where x_l^{α}, y_l^{α} and x_r^{α}, y_r^{α}, are the lower and upper limits of the corresponding α-cuts. Then a distance can be defined as:

$$D[X,Y] = \int_{\alpha=0}^{1} [||x_l^{\alpha}, y_l^{\alpha}|| + ||x_r^{\alpha}, y_r^{\alpha}||] d\alpha \qquad (1)$$

where $||.,.||$ is a distance norm of classical numbers. This distance definition can be extended to n-dimensional ADFVs as:

$$D_n[\mathbf{X},\mathbf{Y}] = \frac{1}{2(n-1)\pi} \int_{\theta_1=0}^{\pi} \cdots \int_{\theta_{n-1}=0}^{\pi} \int_{\alpha=0}^{1} (||\mathbf{x}_l^{\theta\alpha}, \mathbf{y}_l^{\theta\alpha}|| + $$
$$+ ||\mathbf{x}_r^{\theta\alpha}, \mathbf{y}_r^{\theta\alpha}||) d\alpha d\theta_{n-1} \ldots d\theta_1 \qquad (2)$$

where $\mathbf{x}_l^{\theta\alpha}$, $\mathbf{y}_l^{\theta\alpha}$ and $\mathbf{x}_r^{\theta\alpha}$, $\mathbf{y}_r^{\theta\alpha}$ are the lower and upper points that limit the α-cuts of the corresponding 1-d X^{θ} fuzzy numbers, and $||.,.||$ denotes a distance norm between classical vectors. In the following the α-cuts of the X^{θ} fuzzy vectors will be called $\theta\alpha$-cuts and will be symbolized as $\mathbf{X}^{\theta\alpha}$. The use of ADFVs guarantees that every point that belongs to the line segment from $\mathbf{x}_l^{\theta\alpha}$ to $\mathbf{x}_r^{\theta\alpha}$ belongs also to the $\theta\alpha$-cut.

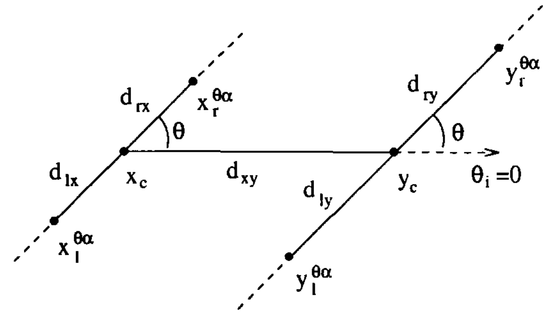


Figure 2: The upper $\mathbf{x}_r^{\theta\alpha}, \mathbf{y}_r^{\theta\alpha}$ and lower $\mathbf{x}_l^{\theta\alpha}, \mathbf{y}_l^{\theta\alpha}$ limits of two $\theta\alpha$-cuts $X^{\theta\alpha}, Y^{\theta\alpha}$, the centers of the ADFVs x_c, y_c and the distances between them.

Let $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ be ADFVs. We can prove that the following distance properties are valid:

- $D_n[\mathbf{X},\mathbf{Y}] = 0 \Leftrightarrow \mathbf{X} = \mathbf{Y}$
- $D_n[\mathbf{X},\mathbf{Y}] = D_n[\mathbf{Y},\mathbf{X}]$
- $D_n[\mathbf{X},\mathbf{Z}] \leq D_n[\mathbf{X},\mathbf{Y}] + D_n[\mathbf{Y},\mathbf{Z}]$

2.3. Euclidean fuzzy distance

Let us choose the Euclidean norm to define a distance between two classical n-dimensional vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ as:

$$d_e^2(\mathbf{x},\mathbf{y}) = (x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2 \quad (3)$$

Then the Euclidean fuzzy distance can be defined by using (2) and (3). When the fuzzy vectors are described by using α-cuts, for a given α and a vector of angles $\theta = (\theta_1, \theta_2, \ldots, \theta_{n-1})$, two points $\mathbf{x}_l^{\theta\alpha}$ and $\mathbf{x}_r^{\theta\alpha}$ are defined, which are the lower and the upper limits of the corresponding $\theta\alpha$-cut. The proposed Euclidean fuzzy distance is the normalized integral of all the distances $d_e^2(\mathbf{x}_l^{\theta\alpha}, \mathbf{y}_l^{\theta\alpha})$ between the lower limits, and the distances $d_e^2(\mathbf{x}_r^{\theta\alpha}, \mathbf{y}_r^{\theta\alpha})$ between the upper limits, for every $\alpha \in [0,1]$ and $\theta_i \in [0, \pi)$, $i = 1, 2, \ldots, n-1$.

Let us symbolize as $d_{lx}^{\theta\alpha}$ the Euclidean distance between the lower limit $\mathbf{x}_l^{\theta\alpha}$ of the $\theta\alpha$-cut and the center $\mathbf{x}_c$ of an ADFV $\mathbf{X}$, as $d_{rx}^{\theta\alpha}$ the Euclidean distance between the upper limit, $\mathbf{x}_r^{\theta\alpha}$ of the $\theta\alpha$-cut and the center $\mathbf{x}_c$ of an ADFV $\mathbf{X}$, and as d_{xy} the distance between the centers of two ADFVs $\mathbf{X}, \mathbf{Y}$. These distances, which can be calculated by using (3), are shown in Figure 2.

It is easy to prove that the distance between two lower limits of two ADFVs α-cuts is equal to:

$$d_e^2(\mathbf{x}_r^{\theta\alpha}, \mathbf{y}_r^{\theta\alpha}) = (d_{lx}^{\theta\alpha} - d_{ly}^{\theta\alpha})^2 +$$
$$+ 2(d_{lx}^{\theta\alpha} - d_{ly}^{\theta\alpha}) d_{xy} \prod_{i=1}^{n-1} \cos(\theta_i) + d_{xy}^2 \qquad (4)$$

where θ_i, $i = 1, 2, \ldots, n-1$ are known angles $\theta_i \in [0, \pi)$. The distance between two upper limits of two ADFVs α-cuts is equal to:

$$d_e^2(\mathbf{x}_r^{\theta\alpha}, \mathbf{y}_r^{\theta\alpha}) = (d_{rx}^{\theta\alpha} - d_{ry}^{\theta\alpha})^2 - \\ +2(d_{rx}^{\theta\alpha} - d_{ry}^{\theta\alpha}) d_{xy} \prod_{i=1}^{n-1} \cos(\theta_i) + d_{xy}^2 \quad (5)$$

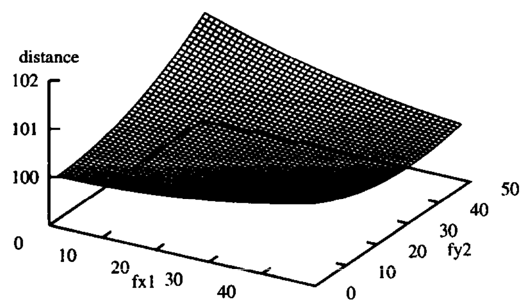
By using (2),(4) and (5) the Euclidean fuzzy distance between two ADFVs X, Y is given by:

$$D_{e_n}[\mathbf{X}, \mathbf{Y}] = d_{xy}^2 + d_{f_{xy}}^2 \quad (6)$$

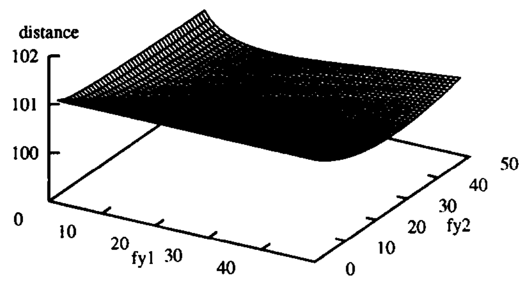
where:

$$d_{f_{xy}}^2 = \frac{1}{2(n-1)\pi} \int_{\theta_1=0}^{\pi} \cdots \int_{\theta_{n-1}=0}^{\pi} \int_{\alpha=0}^{1} [(d_{lx}^{\theta\alpha} - d_{ly}^{\theta\alpha})^2 + \\ +(d_{rx}^{\theta\alpha} - d_{ry}^{\theta\alpha})^2 + \\ +2d_{xy} \prod_{i=1}^{n-1} \cos(\theta_i)(d_{lx}^{\theta\alpha} - d_{ly}^{\theta\alpha} - d_{rx}^{\theta\alpha} + d_{ry}^{\theta\alpha})] d\alpha d\theta_{n-1} \ldots d\theta_1 \quad (7)$$

The above equation shows that the proposed Euclidean fuzzy distance is the classical Euclidean distance between the centers of two ADFVs $\mathbf{X}, \mathbf{Y}$, modified by a factor that depends on the fuzziness that every ADFV holds. The Euclidean fuzzy distance can be considered as a generalized Euclidean distance since equation (7) yields to 0 when the ADFVs are crisp vectors ($d_{lx}^{\theta\alpha} = d_{ly}^{\theta\alpha} = d_{rx}^{\theta\alpha} = d_{ry}^{\theta\alpha} = 0$, $\forall \theta_i, \alpha$). The Euclidean fuzzy distance is also equal to the classical Euclidean distance of the ADFVs centers when the fuzziness of ADFV $\mathbf{X}$ is equal to the fuzziness of ADFV $\mathbf{Y}$ for every angle and α-cut ($d_{lx}^{\theta\alpha} = d_{ly}^{\theta\alpha}$, $d_{rx}^{\theta\alpha} = d_{ry}^{\theta\alpha}$, $\forall \theta_i, \alpha$). Generally, the Euclidean fuzzy distance can be equal to, greater or less than the classical distance of the ADFVs centers, depending on the ADFVs membership functions. Figure 3 shows the distance between two 2-d ADFVs depending on their fuzziness. The ADFVs $\mathbf{X}, \mathbf{Y}$ are assumed to have elliptical α-cuts with axes $f_{x1}^{\alpha}, f_{x2}^{\alpha}$ and $f_{y1}^{\alpha}, f_{y2}^{\alpha}$ respectively, which are reduced linearly from their maximum values $f_{x1}, f_{x2}, f_{y1}, f_{y2}$ for $\alpha = 0$, to zero for $\alpha = 1$. In Figure 3a the distance of the centers is 100, f_{x1}^{α} and f_{y2}^{α} vary from 0 to 50, and $f_{x2}^{\alpha} = \frac{f_{x1}^{\alpha}}{2}$, $f_{y1}^{\alpha} = \frac{f_{y2}^{\alpha}}{2}$. In Figure 3b the distance of the centers is again 100, f_{y1}^{α} and f_{y2}^{α} vary from 0 to 50, and $f_{x1} = 10$, $f_{x2} = 30$. These examples show that the more the fuzziness of the two ADFVs differs, the more their fuzzy Euclidean distance is greater. In special cases, when the fuzziness is not uniformly distributed around the center of a fuzzy set, but it is greater towards the center of the other fuzzy set, the fuzzy Euclidean distance can be less than the classical Euclidean distance.



(a)



(b)

Figure 3: (a) The distance of two ADFVs X, Y depending on their fuzziness f_{x1} and f_{y2} when $f_{x2}^{\alpha} = \frac{f_{x1}^{\alpha}}{2}$, $f_{x1}^{\alpha} = \frac{f_{y2}^{\alpha}}{2}$. (b) The distance of two ADFVs X, Y depending on the fuzziness of Y, f_{y1} and f_{y2} when $f_{x1} = 10$, $f_{x2} = 30$.

3. FUZZY VECTOR MEDIAN DEFINITION AND PROPERTIES

Based on the previously defined distance of ADFVs, we extend the classical definition of the vector median as follows: **Definition 2**: The Fuzzy Vector Median (FVM) of $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ ADFVs is the ADFV $\mathbf{X}_{\text{FVM}}$ such that $\mathbf{X}_{\text{FVM}} \in \{\mathbf{X}_i, i = 1, 2, \ldots, n\}$ and for all $j = 1, 2, \ldots, n$

$$\sum_{i=1}^{n} D_n[\mathbf{X}_{\text{FVM}}, \mathbf{X}_i] \leq \sum_{i=1}^{n} D_n[\mathbf{X}_j, \mathbf{X}_i] \quad (8)$$

A straightforward algorithm to find the FVM of a set of fuzzy vectors is the following:

- for each fuzzy vector $\mathbf{X}_i$ compute the sum of the distances S_i to all other vectors:

$$S_i = \sum_{j=1}^{n} D_n[\mathbf{X}_j, \mathbf{X}_i] \quad (9)$$

- Find k such that S_k is the minimum of S_i, $i = 1, 2, \ldots, n$.
- The Fuzzy Vector Median is X_k.

When the Euclidean fuzzy distance is used the Euclidean FVM is defined. Similarly to the classical vector median, the Euclidean FVM $\mathbf{X}_{\text{FVM}}$ does not minimize the unconditional expression:

$$\mathbf{S}_{e_i} = \sum_{i=1}^{n} D_{e_n}[\mathbf{X}_i, \mathbf{Y}] \tag{10}$$

but, by definition, it minimizes the same expression, when $\mathbf{Y}$ should be one of $\mathbf{X}_i$. The proposed distance of two fuzzy vectors depends on the distance of the crisp fuzzy set centers, and on the fuzziness that every fuzzy vector holds. As a result the Fuzzy Vector Median of a set of fuzzy vectors is affected by the presence of fuzziness. The fuzzy vector, that its center is the classical vector median of the fuzzy vector centers, may be substituted by another fuzzy vector when its fuzziness is different from the fuzziness of its neighbouring vectors.

In the following, the condition which should be valid to take place such a substitution will be found. Let us symbolize as d_{ij} the distance of the centers of two ADFVs $\mathbf{X}_i, \mathbf{X}_j$ that can be calculated by (3), and $d_{f_{ij}}$ the distance depending on the fuzziness given by (7). The classical vector median of the centers of the ADFVs can be found by calculating the sums

$$S_{e_i} = \sum_{j=1}^{n} d_{ij} = \sum_{j=1}^{n} d_e^2(\mathbf{x}_{c_i}, \mathbf{x}_{c_j}) \tag{11}$$

Without loss of generality, we can assume that $S_{e_i} < S_{e_{i+1}}$, $\forall i = 1, 2, \ldots, n-1$, which means that $\mathbf{x}_{c_i}$ is the classical vector median of the centers. Thus, by using (11) the following is also valid:

$$S_{e_{(i+k)}} - S_{e_i} = C_{(i+k)i} > 0 \quad k = 1, \ldots, n-i \tag{12}$$

The ADFV $\mathbf{X}_{i+k}$ will be the FVM if and only if $\mathbf{S}_{e_{(i+k)}} < \mathbf{S}_{e_j}, j = 1, 2, \ldots, n, j \neq i+k$. By using (6) and (12) it can be proven that the above condition is equivalent to:

$$\sum_{j=1}^{n} d_{f_{(i+k)j}} < \sum_{j=1}^{n} d_{f_{ij}} - C_{(i+k)i} \tag{13}$$

The above equation shows that, the vector that corresponds to the classical Vector Median is the most probable candidate to be the Fuzzy Vector Median since $C_{ii} = 0$. It is also more probable to be substituted by its ordered neighbours, and the probability is reduced as the classical distance of the centers C increases.

4. APPLICATIONS

Vector median filters are usually used to remove impulses from noisy color images. In the following, we shall present experimental results when the FVM will be applied on the *lenna* image (256 × 256 pixels), corrupted by impulsive noise and mixed impulsive and Gaussian noise. The FVM, that uses fuzzy Euclidean distance, was applied on a 3 × 3 window. Fuzziness was inserted to the problem by using the information that the neighbouring pixels hold. The chromatic RGB values of the nine pixels of each window was ordered and the average of the differences, between each pixel chromatic value and its two closest values, was used as a measure of the pixel fuzziness. It is obvious that the fuzziness of a pixel changes as the window moves, and when the size of the window is modified. By using this kind of fuzziness, the Fuzzy Vector Median filter removes the impulsive noise and preserves the edges with better performance in comparison with the Vector Median filter. Moreover, FVM reduces the local variances of the filtered image in homogeneous regions.

Let us symbolize as r^o, g^o, b^o the chromatic RGB values of the original image, as r^n, g^n, b^n the values of the noisy image and as r^f, g^f, b^f the values of the filtered image. The Signal to Noise error Ratio (SNR) defined as:

$$\text{SNR} = \frac{(r^o - r^f)^2 + (g^o - g^f)^2 + (b^o - b^f)^2}{(r^o - r^n)^2 + (g^o - g^n)^2 + (b^o - b^n)^2} \tag{14}$$

was used to demonstrate the better performance of FVM versus classical VM. The results of the FVM and VM filters applied on the image *lenna*, corrupted with different values of impulsive and Gaussian noise, are presented in Table 1. It shows that the FVM reduces the error ratio in all cases. The local variances are calculated for every pixel, by using a 3 × 3 window centered on it. Let $\overline{r}, \overline{g}, \overline{b}$ be the average of the RGB values of the pixels that belong in the window. Then the local variance of a pixel is:

$$\sigma^2(k, l) = (r^f - \overline{r})^2 + (g^f - \overline{g})^2 + (b^f - \overline{b})^2 \tag{15}$$

The average of the local variances $\overline{\sigma}$ of all the pixels is presented in Table 2. It shows that the FVM also reduces the local variances in most cases.

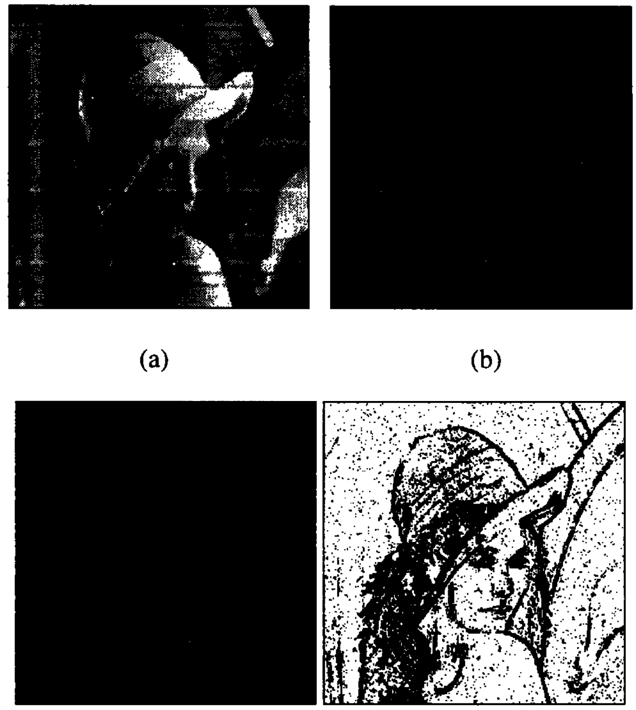
Figure 4a shows the result of the FVM filter applied on a corrupted with mixed impulsive ($p = 0.2$) and Gaussian ($s = 10$) noise. The local variances of the VM and FVM filtered images are shown in Figures 4b and c respectively. Figure 4d shows the differences between the local variances of the FVM filtered images shown in Figure 4b and c. The variances are subtracted and the red channel corresponds to the positive differences, where the FVM local variances are greater. The green channel corresponds to the negative differences, where the VM local variances are greater. The blue channel corresponds to the local variances of the original lenna (edges). It is shown, that there are many cases where the FVM local variances are greater on the edges (pink pixels) something that is desirable and means that the FVM filter preserves the edges better than VM filter does. It is also shown that the VM local variances are greater (green

Table 1: The SNR of the FVM and VM filters applied on the image *lenna*, corrupted with different values of impulsive (percentage of corrupted pixels p=0.1, 0.2) and Gaussian noise (noise standard deviation s=0,10,20).

Impulsive noise (p)	Gaussian noise (s)	FVM (SNR)	VM (SNR)
0.1	0	**0.1850**	0.1853
0.1	10	**0.1901**	0.1910
0.1	20	**0.1997**	0.2004
0.2	0	**0.1067**	0.1070
0.2	10	**0.1125**	0.1134
0.2	20	**0.1283**	0.1293

Table 2: The average of the local variances $\overline{\sigma}$ when the FVM and VM filters are applied on the image *lenna*, corrupted with different values of impulsive (percentage of corrupted pixels p=0.1, 0.2) and Gaussian noise (noise standard deviation s=0,10,20).

Impulsive noise (p)	Gaussian noise (s)	FVM ($\overline{\sigma}$)	VM ($\overline{\sigma}$)
0.1	0	**6.603**	6.604
0.1	10	**8.590**	8.609
0.1	20	11.412	**11.410**
0.2	0	**7.398**	7.417
0.2	10	**9.564**	9.600
0.2	20	**12.751**	12.757



(a) (b) (c) (d)

Figure 4: (a) The FVM filtered image *lenna* corrupted by impulsive (p=0.2) and Gaussian (s=10) noise. (b) The local variances of the VM filtered image. (c) The local variances of the FVM filtered image. (d) The differences between the local variances of the FVM and VM filtered images (FVM-VM: red channel, VM-FVM: green channel, the local variances of the original lenna: blue channel).

pixels) in most cases in homogeneous regions something that is undesirable and is reduced by using FVM filter (red pixels).

5. REFERENCES

[1] N. B. Karayiannis, P. I. Pai. "Fuzzy Vector Quantization Algorithms and their Application in Image Compression", *IEEE Transactions on Image Processing*, Vol. 4, No. 9, pp. 1193-1201, 1995.

[2] F. Russo and G. Ramponi, "Introducing the Fuzzy Median Filter", *in Proceedings of EUSIPCO'94*, pp. 963-966, Edinburgh, UK, September 1994.

[3] X. Yang and P. S. Toh, "Adaptive Fuzzy Multilevel Median Filter", *IEEE Transactions on Image Processing*, Vol. 4, No. 5, pp. 680-682, May 1995.

[4] A. Taguchi, N. Izawa, "Fuzzy Center Weighted Median Filters", *in Proceedings of EUSIPCO'96*, pp. 1721-1724, Trieste, Italy, September 1996.

[5] V. Chatzis and I. Pitas, "Mean and Median of Fuzzy numbers", *in Proceedings of IEEE Workshop on Nonlinear Signal and Image Processing*, pp. 297-300, Neos Marmaras, Greece, June 1995.

[6] J. Astola, P. Haavisto, Y. Neuvo. "Vector Median Filters", *IEEE Proceedings*, Vol.78, No. 4, pp. 678-689, 1990.

[7] S. G. Tzafestas, A. N. Venetsanopoulos. *Fuzzy Reasoning in Information, Decision and Control Systems*, Kluwer Academic Publishers, Dordrecht, 1994.

[8] A. Kaufmann, M. M. Gupta. *Introduction to Fuzzy Arithmetic: Theory and Applications*, Van Nostrand Reinhold, New York, 1985.

[9] R. Kruse, K. D. Meyer. *Statistics with Vague Data*, D. Reidel Publishing Company, Dordrecht, 1987.

[10] I. Pitas, A. N. Venetsanopoulos. *Nonlinear Digital Filters*, Kluwer Academic Publishers, Boston, 1990.

[11] L. H. Tsoukalas, R. E. Uhrig. *Fuzzy and Neural Approaches in Engineering*, Wiley, New York, 1996.