NONLINEAR METHODS IN DIFFERENTIATORS

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ABSTRACT

Differentiation of a signal is required in many applications in the field of signal processing. Good linear differentiators exist which can be used also in the noisy conditions if the noise is not impulsive. In this paper we consider noise corrupted discrete-time measurements whose time derivatives we estimate. We have experimentally evaluated various nonlinear methods and according to our results any of them is superior to the others. In this paper we propose two methods whose performance has been satisfactory in our experiments for differentiating a signal simultaneously corrupted by Gaussian and impulsive type of noise. First of the methods is median prefiltering followed by linear FIR differentiator and the second method is based on robust regression. We also address the problem of second order differentiation.

1. INTRODUCTION

Differentiation is a method to approximate instantaneous rate of change or slope of a signal. There are many important applications in the field of signal processing in which differentiators can be utilized. In biomedical engineering the slope of a signal can be used, e.g., to measure the rate of saturation or second derivative can give important information about the beginning or end of a particular phenomenon. For example differentiation has been used for obtaining the time derivative of left ventricular pressure in [1] and in [2] differentiation has been applied to ECG signal processing.

When we are using digital techniques, we have to sample the analog signal by using some sampling period T. We assume here, in order not to unnecessarily complicate the calculations, that the sampling period is equal to one unit of time, i.e., $T = 1$. Because of sampling we have to construct a discrete-time approximation of the derivative operator in order to be able to calculate the derivative. We have to remember now that the discrete-time samples can be corrupted by noise and try to avoid the weakness of many algorithms designed for this purpose, i.e., the susceptibility to noise.

2. LINEAR DIFFERENTIATORS

One very coarse possibility to approximate the slope of a signal is the first-order difference, i.e., $y(n) = x(n)-x(n-1)$. This approximation assumes linear behavior of the signal between the two consecutive samples. The slope of a straight line between these two points is taken as the derivative estimate. For one thing, this method creates an error because it actually estimates the derivative at a point halfway between the two samples rather than at neither of the two sample points. However, the difference is very easy to calculate and gives nice results when the signal is not corrupted by noise, but this estimate is also extremely sensitive to noise and it has a tendency to amplify the noise so much that the noise totally corrupts the output of the differentiator.

We mentioned above the biological and biomechanical signals as a possible area of applications. These signals usually have low frequency components contaminated by wide-band noise. This is the reason why many low-pass differentiation algorithms have been designed for differentiation of these signals. One such low-pass filter performing both smoothing and differentiation simultaneously has been introduced in [3]. As in many other approaches, also in this one, the goal is quite different from ours, since the interest is very much in the simplicity of the algorithm which of course sets limits to the accuracy of the obtained numerical results.

Because the signals are noisy in most of the real life signal processing applications there is a need for such a differentiator which can estimate the signal slope also under noisy conditions. Many such differentiators have been proposed, one of which is the FIR differentiator introduced in [4]. This differentiator has efficient recursive implementation and it is optimal in attenuation of white Gaussian noise. The impulse response character-

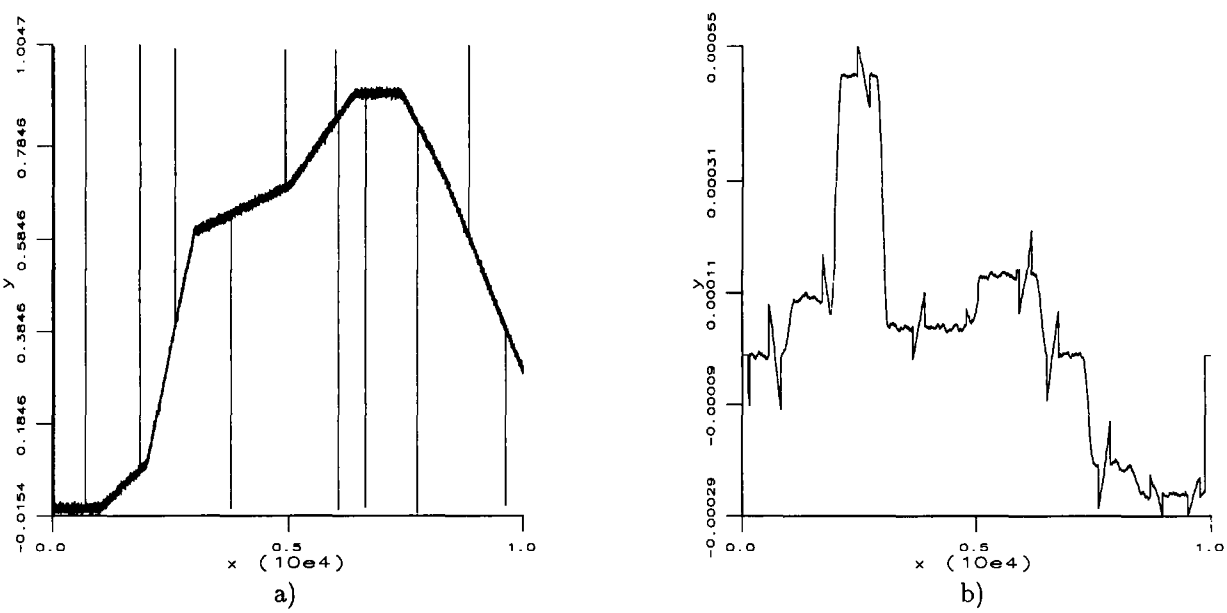


Figure 1: a) Noisy test signal and b) output of FIR differentiator with window length 255.

izing the differentiator is the following

$$h(n) = \frac{6}{(2N+1)(2N+2)}\left(1 - \frac{n}{N}\right), \quad n = 0, \ldots, 2N,$$

where $2N+1$ is the length of the filter window.

This differentiator works well when the type of noise is such that the deviations from the correct values are not very large. In conditions where impulsive type of noise is present the result of linear FIR differentiator deteriorates significantly. In Figure 1 a) Gaussian noise with deviation of 0.005 and some impulses are added to the test signal. From Figure 1 b) it can clearly be seen that FIR differentiator attenuates Gaussian type of noise efficiently, but impulses create very disturbing large deviations to the output of the differentiator. This implies that there exists a need to use nonlinear methods in differentiators when the linear methods fail.

3. LINEAR REGRESSION

Assuming a sliding window of length $2N+1$, we wish to approximate the signal values in the window by a straight line $y = ax + b$. Signal values in the window are $y_1, y_2, \ldots, y_{2N+1}$ and occure at the time instants $x_1, x_2, \ldots, x_{2N+1}$. The slope a of the line $y = ax + b$ is interpreted as the derivative of the signal at the middle point of the window. So we want to study the dependence of a random variable Y on variable X. For this purpose we can use the method of linear regression. In this approach the unknowns a and b are solved by using the principle of least squares, i.e., by minimizing $L = E\{Y - aX - b\}^2$. This square is minimized when the two partial derivatives $\frac{\partial L}{\partial a}$ and $\frac{\partial L}{\partial b}$ are set equal to zero and a and b can then be solved from the two obtained equations (cf. e.g. [5]). This way we obtain

$$a = \frac{E\{XY\} - E\{X\}E\{Y\}}{E\{X^2\} - E\{X\}^2} = \frac{\operatorname{cov}(X,Y)}{\operatorname{var}(X)} \tag{1}$$

and

$$b = E\{Y\} - aE\{X\}. \tag{2}$$

The line of regression of Y on X now obtains the form

$$y = \frac{\operatorname{cov}(X,Y)}{\operatorname{var}(X)}(x - E\{X\}) + E\{Y\}.$$

Because we have now only $2N+1$ samples for calculations, we have to approximate expected values, for example, $E\{XY\} = \frac{1}{2N+1}\sum_{i=1}^{2N+1} x_i y_i$.

As a measure of deviation of the signal values y_i from the predicted counterparts $\hat{y}_i = ax_i + b$ we can use the residual $\hat{e}_i = y_i - \hat{y}_i$ or squared residual $\hat{e}_i^2$. The residual describes how well at each sample point y_i in the window the calculated straight line approximates this sample point. When the samples y_i are noisy the squared residuals $\hat{e}_i^2$ obtain larger values and gross outliers can have very disturbing influence on the estimate. In Figure 2 is an example of a situation where there are two outliers, whose influence is so dominating that the slope of the regression line (–· line) has a value $a = -0.001186514$ instead of the correct value $a = 0.000142857$. So this method obviously needs to

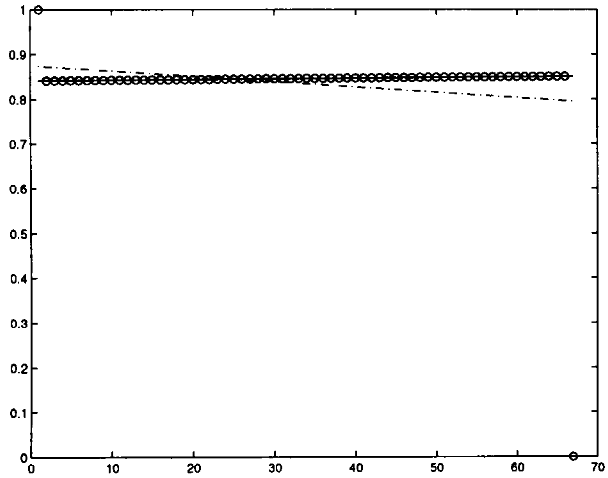


Figure 2: Set of 67 input samples (o) having two outliers at $x = 1$ and $x = 67$. Straight lines are obtained by linear regression of all the samples (—· line) and by linear regression of 65 samples having minimum range (solid line).

be improved in order to eliminate the influence of outliers. We will do this improvement in the next section by rejecting the outliers from the calculation of the expected value by a method proposed there as the second nonlinear method.

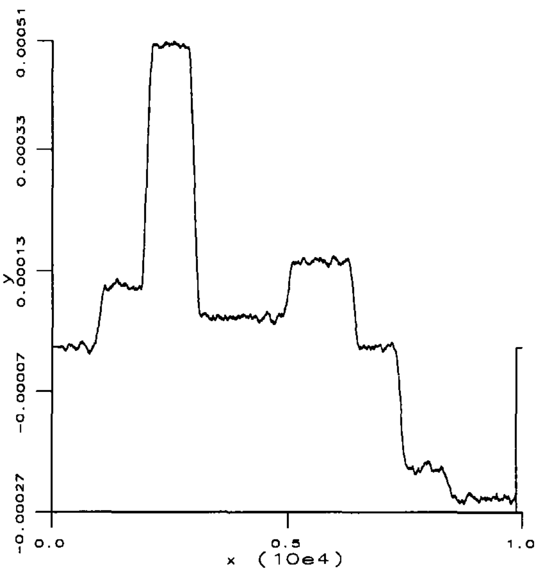
4. NONLINEAR METHODS

We experimented several nonlinear methods for differentiation. Most of our experiments gave good results and we did not find a single method which would have clearly outperformed the others, but several of the methods gave equally good results. In this paper we propose two of those nonlinear methods which performed well also in conditions where there were large errors in the samples, as for example, when the type of noise was impulsive. The two possibilities we concentrate on are nonlinear prefiltering followed by linear filtering and the use of nonlinear expected value estimation method.

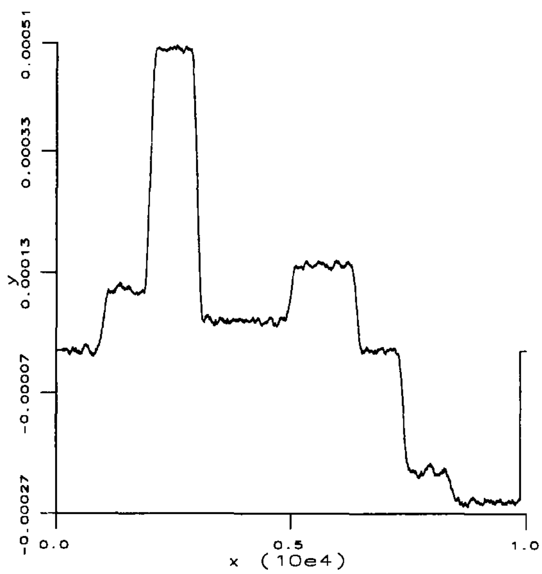
In nonlinear prefiltering we compared the performance of different nonlinear methods in order to find the best prefilter. We also investigated the possibility to use nonlinear postfiltering, but concluded that the results obtained by prefiltering were superior to those obtained by postfiltering. Selection of the prefilter depends of course on the type of noise we want to remove from the signal before applying the FIR differentiator. Because impulsive type of noise corrupted the result of FIR differentiator quite severely, as we saw in the Figure 1 b), we explored several different nonlinear filters which could remove the impulses, but would not distort the signal in other ways so that the FIR differentiator could still operate successfully. In most of the cases very good performance was obtained by using a simple three point median filter. The result of this method applied to the signal in the Figure 1 a) is illustrated in Figure 3 a). There the noisy test signal is first filtered with three point median and after that 255 point FIR differentiator is applied. As can be seen from this Figure the result is very good and essentially the same as obtained in [4] for same input signal without impulses and by using only FIR differentiator. Three point median is a fast operation and so it does not increase computational time of the differentiator very much. Median filtering can also have some drawbacks as described in [6] when the slope of the signal is large compared to noise level. In such a case the output of the median filter is a noisy value causing problems for the FIR differentiator. Typically the differences between different filters were negligible, thus the simple median is a safe choice.

As another good method we propose a nonlinear method for reducing the susceptibility to noise in the signal of linear regression described in the previous section. Thus, we are doing robust regression. In this differentiation method we estimate the expected value in (1) and (2) by using the idea behind the nonlinear WMMR filter [7]. In WMMR filter we select m of the windowed values with the smallest range and weight these samples. The range of a set of values $\{y_1, y_2, \ldots, y_{2N+1}\}$ is defined to be $\max\{|y_i - y_j|, i \neq j, 1 \leq i, j \leq 2N+1\}$. So the selection of the minimum range finds the most condensed concentration of the values and rejects the outliers.

Our method is a combination of linear regression and WMMR filtering. First we calculate an initial estimate with linear regression as described in Section 3. After that we find the smallest range of the squared residuals $\hat{e}_1^2, \hat{e}_2^2, \ldots, \hat{e}_{2N+1}^2$ to obtain the m subindexes defining the samples y_i and x_i taken into consideration in the calculations. By using only these samples we calculate a final estimate by linear regression of only these m samples. So we have modified the WMMR filter in such a way that we filter samples corresponding to the subindexes obtained from the range calculations. The weights are all set equal to $\frac{1}{m}$, i.e., we calculate the mean of the m selected samples. In Figure 2 is shown the result of this approximation (solid line), when $N = 67$ and $m = 65$. The slope $a = 0.000142838$ of this line is very close to the correct one differing only by the last two decimals. In Figure 3 b) is the signal from Figure 1 a) differentiated by this method and



a)



b)

Figure 3: Output of a) FIR differentiator with window length 255 and three point median prefiltering and b) the differentiator based on linear regression and WMMR filter with window length 255 and $m = 253$.

as can be seen the performance of the two proposed methods is quite similar. Naturally, there exists various other techniques for robust regression, which are not considered in this paper.

In Figure 4 is Figure 1 a) filtered similarly as in Figures 1 b), 3 a) and 3 b), but with window length 51 instead of 255. When the window is shortened the impulses start to deteriorate the result of FIR differentiator even further. Also the removal of Gaussian noise is not as good as with longer window length, but very similar in all the three cases in Figure 4. It is important in order to ascertain this fact to notice different scaling of the y-axis in the subfigure a).

5. SECOND ORDER DIFFERENTIATION

Second order differentiation in noisy conditions is even more difficult than first order differentiation, since the amplification of the high frequency noise in linear differentiators grows with the order of the derivative to be estimated. In Figure 5 a) is the signal from Figure 1 a) differentiated twice by FIR differentiator and we can observe that some of the noise spikes exceed the level of correct spikes, which should be in those places where the slope of the signal in Figure 1 a) changes.

As can be seen from Figure 5, both of the nonlinear methods work quite well and the correct spikes can be easily detected by using some threshold value. In Figure 5 b) is the result of differentiation twice by FIR differentiator with median prefiltering and in Figure 5 c) the method based on linear regression and WMMR filtering is applied two times to the initial noisy signal 1 a).

6. CONCLUSIONS

Linear differentiators have a good performance in the presence of Gaussian type of noise, but have problems with impulses. For impulsive type of noise we investigated in this paper the possibilities to use nonlinear differentiators. We found several good nonlinear methods in our experiments and selected two of those giving good results in the case of impulsive noise to illustrate the capabilities of nonlinear methods. Based on our experiments we came to the conclusion that linear methods have good performance in all the other cases, except when the noise is such that the values deviate largely from the correct values.

7. REFERENCES

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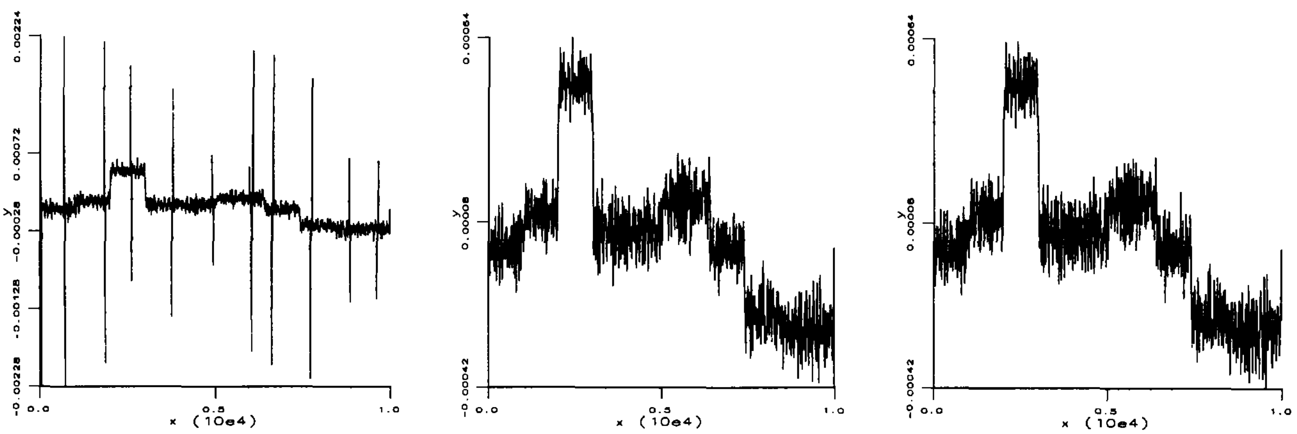
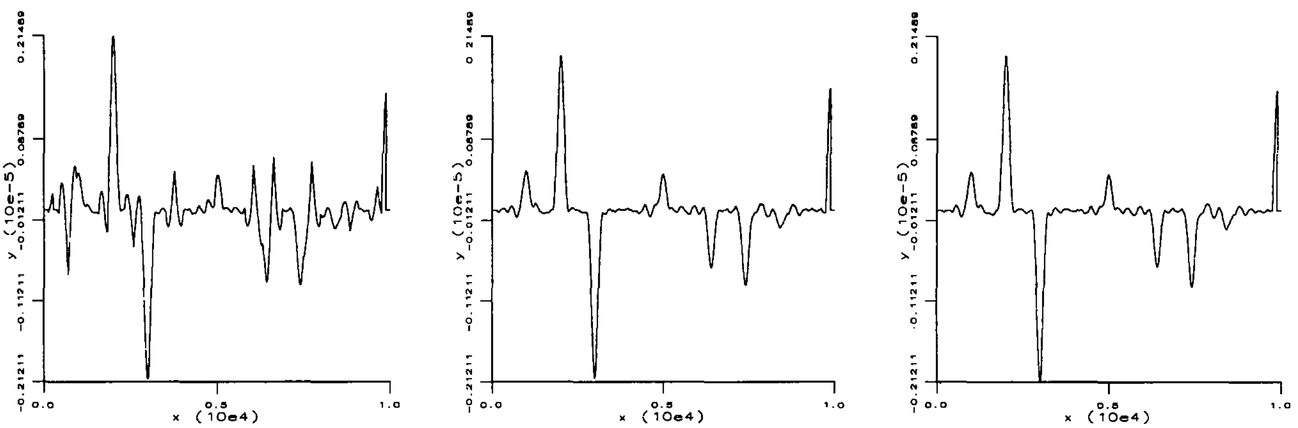


a) b) c)

Figure 4: Output of a) FIR differentiator with window length 51, b) FIR differentiator with window length 51 and three point median prefiltering and c) the differentiator based on linear regression and WMMR filter with window length 51 and $m = 49$. Notice different scales on y-axis.



a) b) c)

Figure 5: Second order derivatives of signal in Figure 1 a) obtained by a) FIR differentiator with window length 255, b) FIR differentiator with window length 255 and three point median prefiltering and c) the differentiator based on linear regression and WMMR filter.

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