IMAGE SHARPENING BY MORPHOLOGICAL FILTERING

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ABSTRACT

In this paper we introduce a class of morphological operators with applications to sharpening digitized grey valued images. We introduce the underlying partial differential equation (PDE) that governs this class of operators. For discrete implementations of the operator class, we show that instances utilizing a parabolic structuring function, have special properties that lead to an efficient implementation and isotropic sharpening behavior.

1. INTRODUCTION

In [1] Kramer et al. define a novel non-linear transformation for sharpening digitized grey valued images. The transformation replaces the grey value at a point by either the minimum or the maximum of the grey values in its neighborhood, the choice depending on which one is closer in value to the original grey value. They show that after a finite number of iterations the resulting image stabilizes, that is every point has become either a local maximum or a local minimum.

In this paper we show that this transformation is an instance of a class of morphological operators which all have sharpening properties. Further, we show that there exists another instance of this class of operators that outperforms the original transformation introduced by Kramer in algorithm order complexity and isotropic sharpening behavior.

2. INTRODUCTION TO MATHEMATICAL MORPHOLOGY

In mathematical morphology [2] the transformation that replaces the grey value at a point by the (weighted) maximum of the grey values in its neighborhood is known as the grey value dilation operator:

$$(f \oplus g)(x) = \bigvee_u [f(u) + g(x-u)]$$

In which function $f(x)$, $f : x \in \mathcal{Z}^2 \mapsto f(x) \in \mathcal{Z}$, is the original image and $g(x)$, $g : x \in \mathcal{Z}^2 \mapsto g(x) \in \mathcal{Z}$, is the structuring function ("neighborhood").

The transformation that replaces the grey value at a point by the (weighted) minimum of the grey values in its neighborhood is known as the grey value erosion operator:

$$(f \ominus g)(x) = \bigwedge_u [f(u) - g(u-x)]$$

Note that in general the grey value dilation operator is extensive: $(f \oplus g)(x) \geq f(x)$ and the grey value erosion operator is anti-extensive: $(f \ominus g)(x) \leq f(x)$. Figure 2a) gives an 1D example of the dilation and erosion operator.

3. STRUCTURING FUNCTIONS

In the remainder of this article we will only consider the following two structuring functions: First, the flat structuring functions as used by Kramer in its original definition of the extreme sharpening operator:

$$c^\rho(x) = \begin{cases} 0 & : x \in \mathcal{S} \\ -\infty & : x \notin \mathcal{S} \end{cases}$$

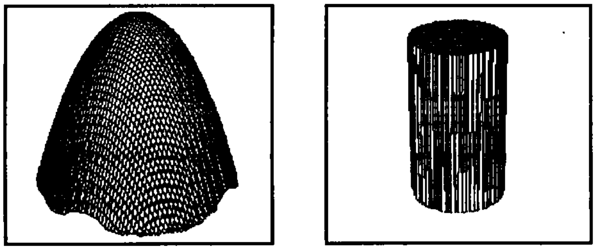
where $\mathcal{S}$ is a disc of radius ρ. Second, the quadratic structuring functions (QSF) as introduced by van den Boomgaard [3]:

$$q^\rho(\mathbf{A})(x) = \rho q(\mathbf{A})(\frac{x}{\rho}) = -\frac{1}{2\rho} < x, \mathbf{A}^{-1}x >$$

where $\mathbf{A}$ is a 2×2 positive definite symmetric matrix. Taking the unity matrix for $\mathbf{A}$ will yield the parabolic structuring function $q^\rho(x) = -\frac{1}{2\rho}x^2$. See for a 2D example of a flat structuring function and a parabolic structuring function figure 1. In [4] van den Boomgaard et al. prove for the class of quadratic structuring functions that:

- Any quadratic structuring function is dimensionally decomposable with respect to dilation.
- The class of quadratic structuring functions contains the unique rotational symmetric structuring function that can be dimensionally decomposed with respect to dilation: $q^\rho(x) = -\frac{1}{2\rho}x^2$.

These properties allow for very efficient algorithms for the dilation operator that has be shown to be independent of the structuring function size [4]. They are typically of order complexity $O(N)$, with N the number of pixels in the original image. Note that for the class of flat structuring functions algorithms for the dilation operator typically have order complexity $O(\rho^2 N)$ and are dependent on the structuring function size ρ.



a) b)

Figure 1: a) Parabolic structuring function $q^{\rho}(x) = -\frac{1}{2\rho}x^2$. b) Flat structuring function $c^{\rho}(x)$.

4. EXTREME SHARPENING OPERATOR CLASS

In this section we give a definition of the extreme sharpening operator class in terms of grey value dilation and grey value erosion operators. Further, we show that iterations of the extreme sharpening operator have sharpening properties.

4.1. Extreme sharpening operator class definition

First, we rephrase the original transformation defined by Kramer in the framework of mathematical morphology:

$$\mathcal{E}[f](x,\rho) = \begin{cases} F^{\oplus}(x,\rho) & : \text{case a.} \\ F^{\ominus}(x,\rho) & : \text{case b.} \\ F(x,0) & : \text{otherwise} \end{cases}$$

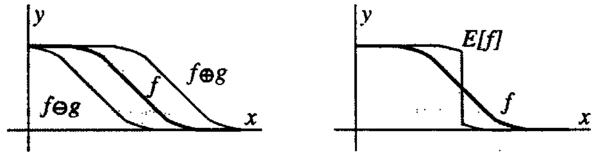
where case a. stands for

$$F^{\oplus}(x,\rho) - F(x,0) < F(x,0) - F^{\ominus}(x,\rho),$$

case b. stands for

$$F^{\oplus}(x,\rho) - F(x,0) > F(x,0) - F^{\ominus}(x,\rho)$$

and where $f(x)$ is the original function, $g(x)$ is the structuring function, x the position, ρ the scale, $F^{\oplus}(x,\rho) = (f \oplus g^{\rho})(x)$, $F^{\ominus}(x,\rho) = (f \ominus g^{\rho})(x)$ and $F(x,0) = F^{\oplus}(x,0) = F^{\ominus}(x,0) = f(x)$. $(f \oplus g)(x)$ and $(f \ominus g)(x)$ are the grey value dilation and grey value erosion operators. This operator class is parameterized by the structuring function $g^{\rho}(x)$. Setting the structuring function $g^{\rho}(x)$ to a flat structuring function $c^{\rho}(x)$ would result in the original definition of Kramer with one modification: Kramer did not consider the special case where $F^{\oplus}(x,\rho) - F(x,0)$ equals $F(x,0) - F^{\ominus}(x,\rho)$. In that case the extreme sharpening operator as defined by Kramer would behave as the grey value dilation operator ($F^{\oplus}(x,\rho)$). In case of a single slope signal ($\nabla^2 f = 0$ everywhere) the application of the operator defined by Kramer would result in a translation of the original signal. Whereas this new definition would preserve the original signal. Figure 2b) gives an 1D example of an instance of the extreme sharpening operator class.



a) b)

Figure 2: a) 1D example of grey value dilation $\oplus$ and erosion operator $\ominus$. b) Extreme sharpening operator.

4.2. Laplacian properties for 1D functions

For every 1D symmetric concave structuring function $g^{\rho}(x)$ the following properties of the extreme sharpening operator class hold:

$$\nabla^2 f(x) < 0 \rightarrow \mathcal{E}[f](x,\rho) > f(x) \tag{1}$$

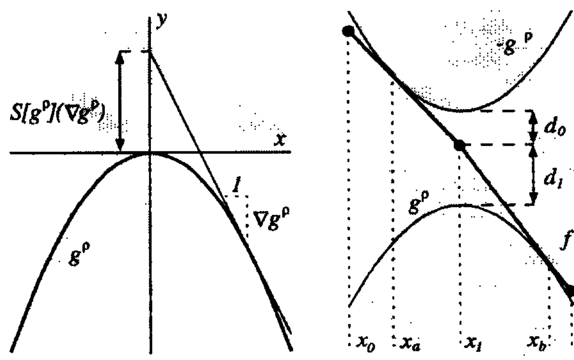
$$\nabla^2 f(x) > 0 \rightarrow \mathcal{E}[f](x,\rho) < f(x), \tag{2}$$

and

$$\nabla^2 f(x) = 0 \rightarrow \mathcal{E}[f](x,\rho) = f(x) \tag{3}$$

where $\nabla^2 f(x)$ is the Laplacian of $f(x)$. A function $g(x)$ is concave if $\forall x_0, x_1$ the line between the points $(x_0, g(x_0))$ and $(x_1, g(x_1))$ is beneath the function $g(x)$.

For symmetric concave structuring functions $g^{\rho}(x)$ we have that for $x > 0$ and increasing x and for $x < 0$ and decreasing x that $\nabla g^{\rho}(x)$ is decreasing. This implies that the intercept of a tangent line of $g^{\rho}(x)$ with the functional axis is higher for $x > 0$ and increasing x and for $x < 0$ and decreasing x, see figure 3a). The intercept with the functional axis is known as the Slope transform $\mathcal{S}[g^{\rho}](\nabla g^{\rho})$, as introduced by Dorst and Van den Boomgaard in [5].



a) b)

Figure 3: a) Intercept with the functional axis (Slope transform). b) Hit-property of dilation and erosion.

See figure 3b), if we take ρ arbitrary small, $\rho \rightarrow 0$, we may linearly interpolate function $f(x)$ between points x_0, x_1 and x_2, as $\nabla^2 f(x) < 0$ point $(x, f(x_1))$ lies above the line between points $(x_0, f(x_0))$ and $(x_2, f(x_2))$. To determine the dilation and erosion value at x_1 we use the hit-property of dilation and by duality erosion. As $\nabla f(x_a) = \nabla - g^{\rho}(x_1 - x_a) = \nabla g^{\rho}(x_1 + x_a)$, $\nabla f(x_b) = \nabla g^{\rho}(x_1 + x_b)$, $\nabla f(x_a) > \nabla f(x_b)$ and $\mathcal{S}[g^{\rho}](\nabla f(x_a)) < \mathcal{S}[g^{\rho}](\nabla f(x_b))$ we have $d_0 < d_1$ setting the extreme sharpening operator value

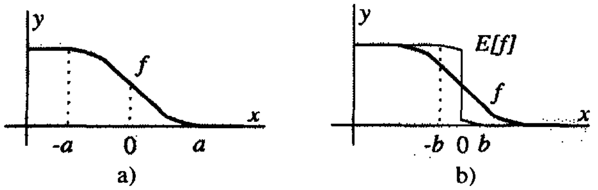


Figure 4: a) Blurred version of original picture. b) One iteration of extreme sharpening operator.

at x_1 to the dilation value $f(x_1) + d_0 = f(x_1) + \mathcal{S}[g^\rho](\nabla f)$, i.e. $F^\oplus(x_1, \rho)$, satisfying property 1. Property 2 is proven by the duality of the erosion operator.

4.3. Laplacian properties for 2D functions

Considering 2D functions properties 1 and 2 do not hold for all points x. As $\nabla^2 f(x) = \partial^2 f/\partial x^2 + \partial^2 f/\partial y^2$, $\nabla^2 f(x) < 0$ holds if both $\partial^2 f/\partial x^2$ and $\partial^2 f/\partial y^2$ are smaller than zero. In this case we have a concave point of $f(x)$ and property 1 still holds. The same is true for property 2 with $\nabla^2 f(x) > 0$ and a convex point of $f(x)$.

If $\partial^2 f/\partial x^2$ and $\partial^2 f/\partial y^2$ do not have the same sign, e.g. a saddle point, which is not convex nor concave, it also depends on the gradient values $\partial f/\partial x$ and $\partial f/\partial y$ whether the dilation or erosion is chosen as the extreme sharpening operator value. For example, if we have $\nabla^2 f(x) < 0$, $\partial^2 f/\partial x^2 < 0$ and $\partial f/\partial y^2 > 0$ we have a point with a local concavity in x and a local convexity in y. The concavity in x would imply that the sharpening operator chooses the dilation value (from x) whereas the convexity in y would imply that the erosion value (from y) is chosen. The choice is made on the highest gradient value, giving the lowest Slope transform value in x or y.

4.4. Sharpening properties

In order to demonstrate the sharpening properties of the extreme sharpening operator class we construct an analytical edge model. Let us suppose that the image we wish to sharpen is the result of passing a black and white picture through a lens and electronic filter which have caused it to become blurred. To retain simplicity in the analysis we shall deal only with one-dimensional pictures. Let $i(x)$ be a function, $i : x \in \mathcal{Z} \mapsto i(x) \in \mathcal{Z}$ of one variable that represents the sampled version of the original picture. The blurred version f(x), $f : x \in \mathcal{Z} \mapsto f(x) \in \mathcal{Z}$ is given by $f(x) = i(x) * h(x)$ where $*$ is the convolution operator and $h(x)$ is the point spread function (PSF), $h : x \in \mathcal{Z} \mapsto h(x) \in \mathcal{Z}$. Let us assume a symmetrical lens, i.e. $h(x) = h(-x)$ with finite aperture, i.e. $h(x) = 0$ for $x < -a$ and $x > a$ Moreover $h(x) \geq 0$ and is decreasing for increasing and decreasing x. For $i(x)$ we take the unity step function:

$$i(x) = \begin{cases} 1 & : x \leq 0 \\ 0 & : x > 0 \end{cases}$$

Note that $\nabla f(x) < 0$ for $x \in [-a, a]$ and as $h(x)$ is decreasing and symmetric that $\nabla^2 f(x) < 0$ for $x \in [-a, 0)$, $\nabla^2 f(x) > 0$ for $x \in (0, a]$ and $\nabla^2 f(x) = 0$ for $x = 0$.

The extreme sharpening operator $E[f]$ will only change $f(x)$ at points x having $\nabla^2 f(x) \neq 0$. Consider the interval $[-a, 0)$ with points x having $\nabla^2 f(x) < 0$, see figure 4a). This interval represents a concave part of the function $f(x)$. Application of one iteration of the extreme sharpening operator with a concave structuring function $g^\rho(x)$ on this interval will result in $F^\oplus(x, \rho)$ at this interval. Note that as $F^\oplus(x, \rho) > f(x)$ at this interval (extensivity property of dilation), that $F^\oplus(x, \rho)$ is also concave (proven in [5]) and that the interval at which points x having $\nabla^2 F^\oplus(x, \rho) < 0$, i.e. $[-b, 0)$, is smaller than the original interval $[-a, 0)$ at which $\nabla^2 f(x) < 0$. So, repeated applications of the extreme sharpening operator on the interval $[-a, 0)$ are well defined and result in an interval at which all points x have function values equal to the maximum function value in the interval $[-a, 0)$.

The same holds for the convex interval $(0, a]$ with points x having $\nabla^2 f(x) > 0$. In this case, repeated applications of the extreme sharpening operator result in an interval at which all points x have function values equal to the minimum function value in the interval.

It can be shown that in case the structuring function g^ρ is rotational symmetric the sharpening properties of the extreme sharpening operator also hold for 2D images.

5. EXTREME SHARPENING OPERATOR: PARTIAL DIFFERENTIAL EQUATION

In this section we introduce the underlying partial differential equation that governs the extreme sharpening operator class. Given $g(x)$ is a concave structuring function and $g^\rho(x) = \rho g(\frac{x}{\rho})$ (umbral scaling) we have:

$$\frac{\partial F^\oplus}{\partial \rho} = \lim_{\Delta\rho \to 0} \frac{F^\oplus(x, \rho + \Delta\rho) - F^\oplus(x, \rho)}{\Delta\rho} \stackrel{1}{=}$$

$$\lim_{\Delta\rho \to 0} \frac{F^\oplus(x, \rho) \oplus g^{\Delta\rho} - F^\oplus(x, \rho)}{\Delta\rho} \stackrel{2}{=} \lim_{\Delta\rho \to 0} \frac{\mathcal{S}[g^{\Delta\rho}](\nabla F^\oplus)}{\Delta\rho} \stackrel{3}{=}$$

$$\lim_{\Delta\rho \to 0} \frac{\Delta\rho \mathcal{S}[g](\nabla F^\oplus)}{\Delta\rho} = \mathcal{S}[g](\nabla F^\oplus)$$

and by duality of the dilation operator, that for $\Delta\rho \to 0$:

$$\frac{\partial F^\ominus}{\partial \rho} = -\mathcal{S}[g](\nabla F^\ominus)$$

where equalities 1 and 3 are proven in [5] and equality 2 is discussed in section 4.2. Using properties 1 and 2 and without considering saddle points in 2D, as discussed in section 4.3 this results in the partial differential equation for the extreme sharpening operator:

$$\frac{\partial \mathcal{E}[f]}{\partial \rho} = \mathrm{sign}[\nabla^2 f]\mathcal{S}[g](\nabla f)$$

where

$$\mathrm{sign}[f](x) = \begin{cases} 1 & : f(x) > 0 \\ -1 & : f(x) < 0 \\ 0 & : \text{otherwise} \end{cases}$$

In the remainder of this section we derive the partial derivative equations of the extreme sharpening operator for the parabolic structuring functions and the flat structuring functions. For parabolic structuring functions $g^\rho(x) = -\frac{1}{2\rho}x^2$, $g(x) = -\frac{1}{2}x^2$ and $\mathcal{S}[g](w) = \frac{1}{2}|w|^2$ the partial differential equation for the extreme sharpening operator becomes:

$$\frac{\partial \mathcal{E}[f]}{\partial \rho} = \frac{\text{sign}[\nabla^2 f]|\nabla f|^2}{2}$$

which is apart from sign$[\nabla^2 f]$ similar to the PDE of the morphological scale-space [3]: $\frac{\partial F}{\partial \rho} = |\nabla F|^2$. For flat structuring functions $g^\rho(x)$, $\mathcal{S}[g](w) = |w|$ the partial differential equation for the extreme sharpening operator becomes:

$$\frac{\partial \mathcal{E}[f]}{\partial \rho} = \text{sign}[\nabla^2 f]|\nabla f|$$

In the next section we will look at the use of both structuring functions $g^\rho(x)$ in case of the application of the extreme sharpening operator in a discrete domain. It will be shown that applications of the extreme sharpening operator for small values of ρ is a numerical difference scheme to solve the partial differential equation of the extreme sharpening operator, in which the stability of the numerical difference scheme depends on the choice of the structuring function, the type of function values and the minimum value ρ that can be set for a structuring function $g^\rho(x)$ in the discrete domain.

6. DISCRETE APPROXIMATION AND EXPERIMENTS

In this section we will present results of applications of the extreme operator using flat structuring functions and parabolic structuring functions in the discrete domain. In the discrete domain we consider function values as numbers given on a grid. We consider two types of function values. The first type is integer function values: for instance, the grey value range $[0, N]$, where N typically equals 256. The second type of function values utilizes a floating point representation. Although the second type is still a discrete type it has advantages over the first type at the cost of more required storage space for sampled function values. We will show for both types of function values that the choice of a quadratic structuring function $q^\rho(x)$ as the $g^\rho(x)$ structuring function for the extreme sharpening operator is favorable over a flat structuring function.

6.1. Applications of the extreme sharpening operator

For accurate results of the extreme sharpening operator in the discrete domain it is necessary to choose the ρ value of the corresponding structuring function $g^\rho(x)$ as small as possible. Reducing the value of ρ increases the number of necessary iterations of the extreme sharpening operator. The image sharpening can be performed with the application of one step of the extreme sharpening operator for a certain (large) value of ρ at the cost of losing image details smaller than the structuring function in the resulting image. We propose the repeated application of the extreme sharpening operator using a small value for ρ.

6.2. Integer function values

When we compare the flat structuring function $c^\rho(x)$ with the parabolic structuring function $q^\rho(x)$ in the discrete domain, where function values are given as integer values on a grid, we notice that the flat structuring function $c^\rho(x)$ can only be as small as possible for $\rho = 1$. The discrete approximation of the disc $\mathcal{S}$ of the flat structuring function $c^\rho(x)$ then equals a diamond (4-connected) or a square (8-connected).

In case of the structuring function $q^\rho(x)$, ρ can be as small as possible: $q^\rho(x)$ is in effect an infinite response filter. But in order to have any effect on an image it has to have a minimum value of $\rho = 1$. For lower values of ρ the extreme sharpening operator will not be able to fully sharpen the image, only up to a maximum slope.

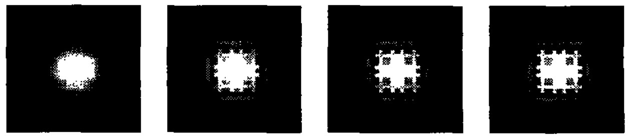


Figure 5: Applications of the extreme sharpening operator for 2, 4 and 8 iterations using a 4-connected flat structuring function with $\rho = 1.0$.

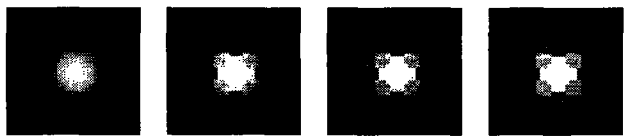


Figure 6: Applications of the extreme sharpening operator for 2, 4 and 8 iterations using a 8-connected flat structuring function with $\rho = 1.0$.

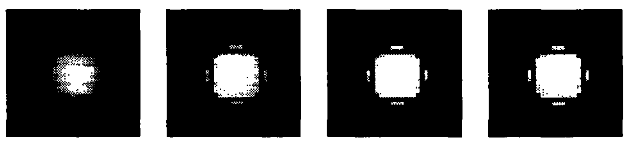


Figure 7: Applications of the extreme sharpening operator for 2, 4 and 8 iterations using a parabolic structuring function $q^\rho(x)$ with $\rho = 1.0$.

In figures 5, 6 and 7 we have depicted the results of the application of the extreme sharpening operator for different numbers of iterations and different structuring functions $g^\rho(x)$ in case of integer function values. The original image is a digitized 2D Gauss function. The desired result of the application of the extreme sharpening operator should be a cylinder. From the results we may conclude that sharpening with a parabolic structuring function $q^\rho(x)$ resembles the desired result better then when a 4-connected or 8-connected flat structuring function is used. This stems from the fact that the discrete approximation of a parabolic structuring function $q^\rho(x)$ is more *isotropic* then the discrete approximation of the disc $\mathcal{S}$ of the 4-connected and

8-connected flat structuring function. But still, the discrete integer valued approximation of $q^{\rho}(x)$ contains (repeating) discretization errors, as noted in [6], which will influence the correctness of the extreme sharpening operator and the stability of the sharpening result after several iterations of the extreme sharpening operator, which can be seen in figure 7.

6.3. Floating point function values

In case of floating point function values given on a grid, the ρ value for the parabolic structuring functions can even be lower than 1 (down to ϵ), which is determined by the floating point precision. The minimal value of ρ for flat structuring functions remains 1.

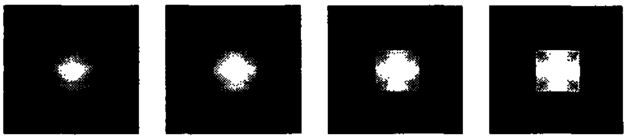


Figure 8: Applications of the extreme sharpening operator for 2, 4 and 8 iterations using a 4-connected flat structuring function with $\rho = 1.0$.

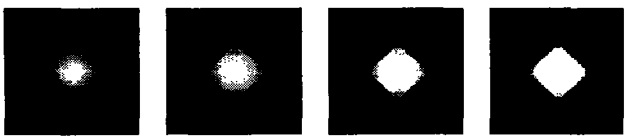


Figure 9: Applications of the extreme sharpening operator for 2, 4 and 8 iterations using a 8-connected flat structuring function with $\rho = 1.0$.

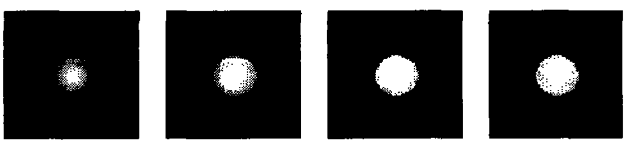


Figure 10: Applications of the extreme sharpening operator for 2, 4 and 8 iterations using a parabolic structuring function $q^{\rho}(x)$ with $\rho = 0.2$.

In figures 8, 9 and 10 we have depicted the results of the application of the extreme sharpening operator for different numbers of iterations and different structuring functions $g^{\rho}(x)$ in case of floating point function values. The original image is again a digitized 2D Gauss function and consequently the desired result of the application of the extreme sharpening operator should be a cylinder. From the results we may conclude that sharpening with a parabolic structuring function $q^{\rho}(x)$ correctly yields the desired result: a cylinder, whereas the sharpening with a 4-connected or 8-connected flat structuring function gives cubic-like figures. Again due to the anisotropic behavior.

6.4. Continuous approximation

In the discrete case the ρ of both the parabolic structuring functions and flat structuring functions has a minimum bound to ensure any sharpening effect. Further research should indicate whether it is possible to use the 1D *union-of-translations* implementation of dilation with a parabolic structuring function as described in [4] to come up with an implementation of the extreme sharpening operator in which case we can choose ρ arbitrary small.

7. CONCLUSIONS

We have introduced a class of morphological operators with applications to sharpen digitized grey value images. We have defined the extreme sharpening operator class in terms of the grey value dilation and erosion operator from mathematical morphology and derived the partial differential equation (PDE) that governs this class of operators. Furthermore, we have shown the sharpening properties of this class of operators given an analytical 1D edge model. We have focused on two instances of this class of operators: one utilizing a flat structuring function and one using a parabolic structuring function. We have shown with experiments in the discrete domain for two types of function values, integer and floating point, that the use of a parabolic structuring function is favorable over a flat structuring function in terms of algorithm complexity and isotropic sharpening behavior.

Acknowledgments

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