THE STABILITY OF BILINEAR AND QUADRATIC FILTERS

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ABSTRACT

We consider stability properties of discrete-time bilinear filters. Simple sufficient conditions are given for bounded-input bounded-output stability (with not necessarily zero initial conditions), l_p stability, and three other important types of stability. In particular, conditions are given under which asymptotically periodic inputs produce asymptotically periodic outputs with the same period. Related results are given for quadratic filters.

1. INTRODUCTION

Bilinear system models arise in a variety of problem settings in the fields of engineering, biology, and economics [1, 2], and it is known [3] that a large class of input-output maps can be realized by state-space bilinear systems. In particular, bilinear systems – together with natural questions concerning their advantages and limitations – are of current interest in connection with signal processing because of the limitations of linear filters.

In this paper we consider the discrete-time "bilinear filter", whose output $y(0), y(1), \ldots$ satisfies the difference equation

$$y(n) = \sum_{i=0}^{N} a_i u(n-i) + \sum_{i=1}^{N} b_i y(n-i) + \sum_{i=0}^{N} \sum_{j=1}^{N} c_{i,j} u(n-i) y(n-j), \quad n \geq 0 \tag{1}$$

in which the a_i, b_i, and $c_{i,j}$ are real coefficients, $u(0)$, $u(1), \ldots$ is the input sequence, $y(-N), \ldots, y(-1)$ and $u(-N), \ldots, u(-1)$ are initial values, and N is a positive integer. The initial values and the elements of the input and output sequences are real numbers. In [4] it is shown that the input-out maps of the members of a large class of single-input single-output bilinear systems described by state-space equations are governed by an equation of the form (1).

There are many worthwhile questions that can be asked about the way in which (1) takes inputs into outputs. For example, it is of interest to know conditions under which bounded inputs produce bounded outputs. One such set of conditions is given in [5], but one of the conditions there is that all of the initial values of the output are zero. In this paper we show that the condition concerning initial values is not needed. More importantly, in Section 2 we give simple conditions under which (1) has the additional stability properties that

(i) If $1 \leq p < \infty$ and the input sequence belongs to the set l_p (see Section 2.1), then the output sequence also belongs to l_p.

(ii) $u(n) \to 0$ as $n \to \infty$ implies $y(n) \to 0$ as $n \to \infty$.

(iii) $y_1(n) - y_2(n) \to 0$ as $n \to \infty$ whenever $u_1(n) - u_2(n) \to 0$ as $n \to \infty$, where y_1 is the output corresponding to the input u_1, and y_2 is the output corresponding to the input u_2.

(iv) If u is asymptotically periodic with some period T, by which we mean that $u(n) = u_p(n) + u_0(n)$ for $n \geq 0$, where $u_p(n) = u_p(n+T)$ for $n \geq 0$ and $u_0(n) \to 0$ as $n \to \infty$, then the output y is asymptotically periodic with period T.

Related results are given in the appendix for quadratic filters. Results of this kind contribute to the construction of an analytical basis for the use of nonlinear filters.

2. BILINEAR FILTER STABILITY RESULTS

2.1. Notation, Definitions, and an Assumption

Let S denote the set of real sequences $s(0), s(1), \ldots$. For $1 \leq p < \infty$, let l_p be the subset of all such sequences

s such that $\sum_{n=0}^{\infty} |s(n)|^p < \infty$. The usual l_p norm is denoted by $\|\cdot\|_{l_p}$. We use l_∞ to denote the set of bounded elements of S, and $\|\cdot\|_\infty$ stands for the usual sup norm.

Let Z^{-1} denote inverse z-transform operator defined on the set of z-transforms of z-transformable elements of S.

Throughout Section 2 we assume that $(1 - \sum_{i=1}^{N} b_i z^{-i}) \neq 0$ for $|z| \geq 1$. This means that we are assuming the stability in a standard sense of the system governed by (1) when $c_{i,j} = 0$ for $0 \leq i \leq N$ and $1 \leq j \leq N$.

In our results, Theorems 1–4 below, we refer to a set U. This set is described as follows. Given a $v \in S$ together with real numbers $v(-N), \ldots, v(-1)$, we say that $(v, v(-N), \ldots, v(-1))$ belongs to U if

$$\|h\|_{l_1} \cdot \sup_{k \geq -N} |v(k)| \cdot \sum_{i=0}^{N} \sum_{j=1}^{N} |c_{i,j}| < 1 \qquad (2)$$

where $h := Z^{-1}\left\{(1 - \sum_{i=1}^{N} b_i z^{-i})^{-1}\right\}$. Of course when $(u, u(-N), \ldots, u(-1)) \in U$, (2) gives a bound on the magnitude of the elements of the input sequence and the magnitude of the corresponding initial conditions. [1] Proofs are omitted in this paper.

2.2. Results

In order to simplify the statement of Theorem 1 we define constants α and β as follows:

$$\alpha = \|h\|_{l_1} \cdot \sup_{k \geq -N} |u(k)| \cdot \sum_{i=0}^{N} |a_i| + r\|h\|_{l_1} \cdot$$

$$\sup_{k \geq -N} |u(k)| \cdot \sum_{i=0}^{N} \sum_{j=1}^{N} |c_{i,j}| + \|I\|_\infty$$

and

$$\beta = \|h\|_{l_1} \cdot \sup_{k \geq -N} |u(k)| \cdot \sum_{i=0}^{N} \sum_{j=1}^{N} |c_{i,j}|$$

where I is the inverse z-transform of

$$\left(\sum_{i=1}^{N} \sum_{j=0}^{i-1} b_i z^{-j} y(j-i) \right) \cdot \left(1 - \sum_{i=1}^{N} b_i z^{-i} \right)^{-1}$$

and $r = \sup_{k=1,\ldots,N} \{|y(-k)|\}$.

[1] $h \in l_1$ by our assumption that $(1 - \sum_{i=1}^{N} b_i z^{-i}) \neq 0$ for $|z| \geq 1$.

Theorem 1: Assume that $(u, u(-N), \ldots, u(-1))$ belongs to U, and suppose that $y(0), y(1), \ldots$ satisfy (1). Then $|y(n)| \leq \alpha\,(1-\beta)^{-1}$, $n \geq 0$.

This theorem gives conditions under which bounded inputs in (1) produce bounded outputs. It tells us that the output is bounded whenever $u \in \ell_\infty$ and

$$\|h\|_{l_1} \cdot \|u\|_\infty \cdot \sum_{i=0}^{N} \sum_{j=1}^{N} |c_{i,j}| < 1.$$

The theorem yields the result given in [5] which concerns the case in which the initial values of the output are zero.[2]

Theorem 2: Assume that $(u, u(-N), \ldots, u(-1))$ belongs to U, let $y(0), y(1), \ldots$ satisfy (1), and let $p \geq 1$. Then $u \in l_p$ implies that $y \in l_p$.

Now we consider the outputs that correspond to input sequences u_1 and u_2 whose difference converges to zero. In the following theorem, S_0 stands for the set of elements of S that converge to zero.

Theorem 3: Let sequences $y_1(0), y_1(1), \ldots$ and $y_2(0), y_2(1), \ldots$ be given by

$$y_1(n) = \sum_{k=0}^{n} h(n-k) \sum_{i=0}^{N} a_i u_1(k-i) +$$

$$\sum_{k=0}^{n} h(n-k) \sum_{i=0}^{N} \sum_{j=1}^{N} c_{i,j} u_1(k-i) y_1(k-j) + z_1(n)$$

and

$$y_2(n) = \sum_{k=0}^{n} h(n-k) \sum_{i=0}^{N} a_i u_2(k-i) +$$

$$\sum_{k=0}^{n} h(n-k) \sum_{i=0}^{N} \sum_{j=1}^{N} c_{i,j} u_2(k-i) y_2(k-j) + z_2(n)$$

for $n \geq 0$, where $(u_1, u_1(-N), \ldots, u_1(-1))$ and $(u_2, u_2(-N), \ldots, u_2(-1))$ belong to U, and $z_1, z_2 \in S_0$. Then $u_1(n) - u_2(n) \to 0$ as $n \to \infty$ implies that $y_1(n) - y_2(n) \to 0$ as $n \to \infty$.

Regarding our bilinear system (1), and using a relation in the omitted proof of Theorem 1, Theorem 3 gives conditions under which the difference $y_1 - y_2$ of two output sequences converges to zero whenever the difference $u_1 - u_2$ of the two corresponding input sequences converges to zero. Since $y_2(n) \to 0$ as $n \to \infty$

[2] More precisely, our theorem provides a somewhat stronger result even for that case.

when u_2 is the zero sequence, we also have the following.

Corollary: Regarding (1), suppose that $(u, u(-N), \ldots, u(-1))$ belongs to U. Then $u \in S_0$ implies that $y \in S_0$.[3]

Now we consider inputs to our bilinear system (1) that are asymptotically periodic. Let T be a positive integer.

Theorem 4: Consider (1). Suppose that $(u, u(-N), \ldots, u(-1))$ belongs to U, and that $u(n) = u_p(n) + u_0(n),\ n \geq 0$ where $u_p(n) = u_p(n+T),\ n \geq 0$ and $u_0(n) \to 0$ as $n \to \infty$. Then the output y satisfies $y(n) = y_p(n) + y_0(n),\ n \geq 0$ where $y_p(n) = y_p(n+T),\ n \geq 0$ and $y_0(n) \to 0$ as $n \to \infty$.

In other words, when $(u, u(-N), \ldots, u(-1))$ belongs to U, asymptotically periodic inputs produce asymptotically periodic outputs with the same period.

2.3. Quadratic Filters

The techniques used in our omitted proofs are useful also in connection with related problems that are "more nonlinear." In particular, related results are given in the appendix for the discrete-time "quadratic filter" whose output $y(0), y(1), \ldots$ satisfies

$$y(n) = \sum_{i=0}^{N} a_i u(n-i) + \sum_{i=1}^{N} b_i y(n-i) + \sum_{i=1}^{N} \sum_{j=1}^{N} c_{i,j} y(n-i) y(n-j), \quad n \geq 0 \tag{3}$$

in which the a_i, b_i, and $c_{i,j}$ are real coefficients, $u(0)$, $u(1), \ldots$ is the input sequence, $y(-N), \ldots, y(-1)$ and $u(-N), \ldots, u(-1)$ are initial values, and N is a positive integer. The initial values and the elements of the input and output sequences are real numbers, as in (1). In [7] conditions are presented under which bounded inputs to quadratic filters produce bounded outputs.[4] There too it is assumed that the initial values of the output are zero. This is a significant restriction, because it leaves open the possibility that the filter might not be bounded-input bounded-out stable for even nonzero initial values that are arbitrarily small in magnitude. In the appendix we show that the condition concerning initial values is not needed, in the sense that small values of the magnitudes of the initial conditions can be accommodated by making a small reduction in the bound on the allowed inputs. More importantly, in the appendix we give simple conditions (on the coefficients, inputs, and initial values) under which (3) has the additional stability properties that (ii), (iii), and (iv) of Section 1 are met.

[3] Another result along these lines is this: Under the hypotheses of the corollary, $y(n)$ approaches a finite limit as $n \to \infty$ whenever $u(n)$ approaches a limit as $n \to \infty$. This follows from a direct modification of the proof of Theorem 3, using the fact that the set of elements x of ℓ_∞ such that x approaches a limit is a closed subset of ℓ_∞.

[4] There is a difference between (5) and the model in [7]. There $a_i = 0$ for $i > 1$. We have added the additional terms because their presence leads to a more useful filter.

3. REFERENCES

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4. APPENDIX: QUADRATIC FILTER STABILITY RESULTS

4.1. Notation, Definitions, and Assumptions

Throughout this appendix S, l_1, l_∞, and Z^{-1} are as described in Section 2.1.

Here too we assume that $(1 - \sum_{i=1}^{N} b_i z^{-i}) \neq 0$ for $|z| \geq 1$. This means that we are assuming the stability in a standard sense of the system governed by (3) when $c_{i,j} = 0$ for all i and j. Also, assume that $a_i \neq 0$ for some i and that $c_{i,j} \neq 0$ for some i and some j. In the interest of simplicity later, let

$$\alpha := \left(\|h\|_{l_1} \cdot \sum_{i=1}^{N} |a_i| \right)^{-1}$$

and let β be any positive number such that

$$\beta < \left(\|h\|_{l_1} \cdot \sum_{i=1}^{N} \sum_{j=1}^{N} |c_{i,j}| \right)^{-1}$$

where $h := Z^{-1} \left\{ (1 - \sum_{i=1}^{N} b_i z^{-i})^{-1} \right\}$. [5]

Let γ be any extended real number such that $\gamma > 4$. In two of our results, Theorems 5 and 7 below, we refer to sets Y and U. The set Y is the set of $(y(-N), \ldots, y(-1))$ such that $\|I\|_\infty \leq \beta/\gamma$ and

$$|y(k)| \leq \frac{1}{2}\, \beta \text{ for } k = -N, \ldots, -1,$$

where

$$I = Z^{-1} \left\{ \left(\sum_{i=1}^{N} \sum_{j=0}^{i-1} b_i z^{-j} y(j-i) \right) \cdot \left(1 - \sum_{i=1}^{N} b_i z^{-i} \right)^{-1} \right\}.$$

The condition that $(y(-N), \ldots, y(-1)) \in Y$ is satisfied for any $\gamma > 4$ for sufficiently small $|y(-N)|, \ldots, |y(-1)|$.

The set U is described as follows. Given a $v \in S$ together with real numbers $v(-N), \ldots, v(-1)$, we say that $(v, v(-N), \ldots, v(-1))$ belongs to U if

$$\sup_{k \geq -N} |v(k)| \leq \left(\frac{1}{4} - \frac{1}{\gamma}\right) \alpha\, \beta. \tag{4}$$

Of course when $(u, u(-N), \ldots, u(-1)) \in U$, (4) gives a bound on the magnitude of the elements of the input sequence and the magnitude of the corresponding initial conditions.

[5] As in Section 2.1, $h \in l_1$ by our assumption that $(1 - \sum_{i=1}^{N} b_i z^{-i}) \neq 0$ for $|z| \geq 1$.

4.2. Results

Theorem 5: Assume that $(u, u(-N), \ldots, u(-1))$ belongs to U, that $(y(-N), \ldots, y(-1))$ belongs to Y, and that $y(0), y(1), \ldots$ satisfies (3). Then $|y(n)| \leq \frac{1}{2}\, \beta$ for all $n \geq 0$.

This theorem gives conditions under which bounded inputs in (3) produce bounded outputs.

An inspection of the omitted proof [6] shows that the theorem holds also if β is described instead by

$$\beta = \left(\|h\|_{l_1} \cdot \sum_{i=1}^{N} \sum_{j=1}^{N} |c_{i,j}| \right)^{-1}.$$

This is the case we have in mind in Section 2.3 in our comparison with the result in [7].[7]

Now we consider the outputs that correspond to input sequences u_1 and u_2 whose difference converges to zero. In the following theorem, and as in Section 2.2, S_0 stands for the set of elements of S that converge to zero.

Theorem 6: Let sequences $y_1(0), y_1(1), \ldots$ and $y_2(0), y_2(1), \ldots$ satisfy

$$y_1(n) = \sum_{k=0}^{n} h(n-k) \sum_{i=0}^{N} a_i u_1(k-i) + \sum_{k=0}^{n} h(n-k) \sum_{i=1}^{N} \sum_{j=1}^{N} c_{i,j} y_1(k-i) y_1(k-j) + z_1(n), \quad n \geq 0 \tag{5}$$

and

$$y_2(n) = \sum_{k=0}^{n} h(n-k) \sum_{i=0}^{N} a_i u_2(k-i) + \sum_{k=0}^{n} h(n-k) \sum_{i=1}^{N} \sum_{j=1}^{N} c_{i,j} y_2(k-i) y_2(k-j) + z_2(n), \quad n \geq 0, \tag{6}$$

where $z_1, z_2 \in S_0$ and $|y_i(n)| \leq \frac{1}{2}\, \beta$ for all $n \geq -N$ and $i = 1, 2$. Then $u_1(n) - u_2(n) \to 0$ as $n \to \infty$ implies that $y_1(n) - y_2(n) \to 0$ as $n \to \infty$.

Regarding our system (3), and using a relation in the omitted proof of Theorem 5, Theorem 6 (together

[6] Proofs of the theorems in this appendix are given in [6].

[7] We have a somewhat stronger result even for the $\gamma = \infty$ case (in which the initial values of the output are assumed to be zero).

with Theorem 5) gives conditions under which the difference $y_1 - y_2$ of two output sequences converges to zero whenever the difference $u_1 - u_2$ of the two corresponding input sequences converges to zero.

A proof similar to the proof of Theorem 6 given in [6] establishes that y in (5) belongs to S_0 when $u \in S_0$, $(u, u(-N), \ldots, u(-1))$ belongs to U and $(y(-N), \ldots, y(-1)) \in Y$.

Now we consider inputs to our system (3) that are asymptotically periodic. Let T be a positive integer.

Theorem 7: Consider (3). Suppose that $(u, u(-N), \ldots, u(-1))$ belongs to U, that $(y(-N), \ldots, y(-1))$ belongs to Y, and that $u(n) = u_p(n) + u_0(n)$, $n \geq 0$ where $u_p(n) = u_p(n+T)$, $n \geq 0$ and $u_0(n) \to 0$ as $n \to \infty$. Then the output y satisfies $y(n) = y_p(n) + y_0(n)$, $n \geq 0$ where $y_p(n) = y_p(n+T)$, $n \geq 0$ and $y_0(n) \to 0$ as $n \to \infty$.

In other words, when $(u, u(-N), \ldots, u(-1))$ belongs to U and $(y(-N), \ldots, y(-1))$ belongs to Y, asymptotically periodic inputs produce asymptotically periodic outputs with the same period.