ON THE EXACT INVERSE AND THE pTH ORDER INVERSE OF CERTAIN NONLINEAR SYSTEMS

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ABSTRACT

This paper presents two theorems for the exact inversion and the pth order inversion of a wide class of causal, discrete-time, nonlinear systems. The nonlinear systems we consider are described by the input-output relationship $y(n) = g[x(n)] + f[x(n-1), y(n-1)]$, where $g[\cdot]$ and $f[\cdot,\cdot]$ are causal, discrete-time and nonlinear operators and the inverse function $g^{-1}[\cdot]$ exists. The exact inverse of such systems is given by $z(n) = g^{-1}\left[u(n) - f[z(n-1), u(n-1)]\right]$. Similarly, the pth order inverse is given by $z(n) = g_p^{-1}\Big[u(n) - f[z(n-1), u(n-1)]\Big]$ where $g_p^{-1}[\,\cdot\,]$ is the pth order inverse of $g[\,\cdot\,]$.

1. INTRODUCTION

Inversion of nonlinear systems is not a trivial task in most situations. Not all nonlinear systems possess an inverse and many nonlinear systems admit an inverse only for a certain subset of input signals. For these reasons, Schetzen has developed the theory of the pth order inverse of a nonlinear system whose input-output relation can be represented using Volterra series expansions [7], [8]. The pth order inverse of a nonlinear system H is defined as the pth order system which, connected in cascade with H, results in a system whose Volterra kernels from the second up to the pth order are zero. A pth order system is one in which all the Volterra kernels of order greater than p are zero. The definition of the pth order inverse was relaxed in [6] by allowing the inverse system to possess non-zero Volterra operators of order greater than p. These operators do not affect the first p Volterra operators of the cascade system and in [6] they are used in order to derive more simple and computationally efficient expressions for the inverse system. However, because of the presence of higher order components, the definition of the pth order inverse in [6] does not result in a unique inverse system. Both the approaches of [6] and [7] lead to the same result for the existence and the stability of the pth order inverse. If the linear part (*i.e.* the first order Volterra kernel) of the system H admit a Bounded Input Bounded Output stable inverse, then the pth order inverse exists, it is BIBO stable and it depends only from the first p Volterra operators of H. In this paper, we accept the instability or the input dependent stability of the resulting system in order to obtain the exact inverse of a particular class of discrete-time causal nonlinear systems. The systems we are interested in are described by the following input-output relationship:

$$y(n) = g[x(n)] + f[x(n-1), y(n-1)] \qquad (1)$$

where $g[\cdot]$ and $f[\cdot,\cdot]$ are discrete-time causal nonlinear operators. Our main contribution is the derivation of an expression for the exact inverse of the class of systems described by (1). The exact inverse of the system in (1) may not exist or may not be stable for certain input signals. However, even if the exact inverse cannot be trivially derived, a more efficient realization of the pth order inverse may be obtained. The efficient realization we propose is derived by the use of a nonlinear feedback.

The rest of this paper is organized as follows. The inverse of the system in (1) is introduced in Section 2. An efficient pth order inverse is derived in Section 3. Section 4 presents some experimental results that confirm the usefulness of these inversion theorems. Concluding remarks regarding the stability of the filters that results from the inversion procedure are discussed in Section 5.

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2. THE INVERSE OF CERTAIN NONLINEAR SYSTEMS

In all our discussions we assume causal signals, *i.e.*, all the signals are identically zero for time indices less than zero. The following theorem show how to evaluate the exact inverse of the system in (1).

Theorem 1 *Let $g[\cdot]$ and $f[\cdot,\cdot]$ be causal nonlinear discrete operators and let the inverse operator $g^{-1}[\cdot]$ exist. Then, the system described by the input-output relationship*

$$z(n) = g^{-1}\Big[u(n) - f\big[z(n-1), u(n-1)\big]\Big] \qquad (2)$$

is the exact inverse of the system in (1).

Proof: We demonstrate first that the system in (2) is the post-inverse of (1), *i.e.*, a cascade interconnection of the system in (1) followed by the system in (2) results in an identity system. We proceed by mathematical induction. Let $x(n)$ and $y(n)$ represents the input and output signals, respectively, of the system in (1). To prove the theorem using induction, we assume that

$$z(n-i) = x(n-i) \qquad \forall i > 0. \qquad (3)$$

We must now show using (3) that

$$z(n) = x(n) \qquad (4)$$

when $u(k) = y(k)$ for $k \leq n$. Now,

$$\begin{aligned} z(n) &= g^{-1}\left[y(n) - f\big[z(n-1), y(n-1)\big]\right] \\ &= g^{-1}\left[g\big[x(n)\big] + f\big[x(n-1), y(n-1)\big] + \right. \\ &\qquad \left. - f\big[z(n-1), y(n-1)\big]\right]. \end{aligned} \qquad (5)$$

By substituting $z(n-i) = x(n-i)$ from (3) into (5), it follows in a straightforward manner that $z(n) = x(n)$. We can prove in a similar manner that the system in (2) is also the pre-inverse of the system in (1), *i.e.*, a cascade interconnection of the system in (2) followed by the system in (1) results in an identity system. This completes the proof.

Example 1: The inverse of the bilinear system

$$\begin{aligned} y(n) = x(n) + \sum_{i=1}^{N-1} a_i x(n-i) + \sum_{i=1}^{N-1} b_i y(n-i) + \\ \sum_{i=1}^{N-1}\sum_{j=1}^{N-1} c_{ij} x(n-i) y(n-j) \end{aligned} \qquad (6)$$

is the bilinear system

$$\begin{aligned} z(n) = u(n) - \sum_{i=1}^{N-1} b_i u(n-i) - \sum_{i=1}^{N-1} a_i z(n-i) + \\ - \sum_{i=1}^{N-1}\sum_{j=1}^{N-1} c_{ij} z(n-i) u(n-j). \end{aligned} \qquad (7)$$

3. pTH ORDER INVERSES

Since the inverse system $g^{-1}[\cdot]$ of Theorem 1 may not always exist or may not be easy to derive, we now consider the existence of the pth order inverses of the same class of systems as before.

Theorem 2 *Let $g[\cdot]$ and $f[\cdot,\cdot]$ be causal discrete-time nonlinear operators with convergent Volterra series expansion with respect to all the arguments. Moreover, let the pth order inverse $g_p^{-1}[\cdot]$ of the system $g[\cdot]$ exist. Then a pth order inverse of the causal discrete-time nonlinear system described in (1) is given by the following input-output relationship*

$$z(n) = g_p^{-1}\Big[u(n) - f\big[z(n-1), u(n-1)\big]\Big]. \qquad (8)$$

Proof: As was the case for Theorem 1, we first show that the system in (8) is the pth order post-inverse of the system in (1). Using the same variables as in the derivation of Theorem 1, we express $z(n)$ as

$$\begin{aligned} z(n) &= g_p^{-1}\Big[y(n) - f\big[z(n-1), y(n-1)\big]\Big] \\ &= g_p^{-1}\Big[g\big[x(n)\big] + f\big[x(n-1), y(n-1)\big] + \\ &\qquad - f\big[z(n-1), y(n-1)\big]\Big]. \end{aligned} \qquad (9)$$

We proceed by mathematical induction. We assume that, for any i greater than zero, the output $z(n-i)$ differs from $x(n-i)$ only by $T_p(n-i)$, a term whose Volterra series expansion in $x(n)$ contains only kernels of order larger than p, *i.e.*,

$$z(n-i) = x(n-i) + T_p(n-i) \qquad \forall i > 0. \qquad (10)$$

We have to prove that the Volterra series expansion of $z(n) - x(n)$ have zero kernels of order up to p. Since $f[\cdot,\cdot]$ admits a convergent Volterra series expansion, we have from (10) that the Volterra series expansion of the difference $f\big[x(n-1), y(n-1)\big] - f\big[z(n-1), y(n-1)\big]$ contains only kernels of order greater than p, *i.e.*,

$$f\big[x(n-1), y(n-1)\big] - f\big[z(n-1), y(n-1)\big] = 0 + T_p'(n), \qquad (11)$$

where the Volterra kernels of $T_p'(n)$ up to order p are zero. Substituting (11) in (9), we get

$$z(n) = g_p^{-1}\left[g[x(n)] + T_p'(n)\right]. \qquad (12)$$

The pth order inverse of the operator $g[\cdot]$ derived in [7] is given by a pth order truncated Volterra series whose kernels depend only on the first p kernels of the Volterra series expansion of $g[\cdot]$. The pth order inverse derived in [6] may have Volterra kernels of order greater than p. However, the inverse still has a Volterra series expansion with finite order of nonlinearity, and it depends only on the first p kernels of the Volterra series expansion of $g[\cdot]$. Consequently, it immediately follows from (12) that

$$z(n) = x(n) + T_p(n) \qquad (13)$$

and that the system in (8) is the pth order post-inverse of the system in (1). We can prove in a similar manner that it is also a pre-inverse of the system in (1).

Example 2: We wish to derive a pth order inverse for the second order Volterra filter given by the following expression:

$$y(n) = \sum_{i=0}^{N-1} a_i x(n-i) + \sum_{i=0}^{N-1}\sum_{j=i}^{N-1} b_{ij} x(n-i)x(n-j). \qquad (14)$$

Let

$$g\left[x(n)\right] = a_o x(n) + x(n)\sum_{j=0}^{N-1} b_{0j} x(n-j) \qquad (15)$$

and

$$f\left[x(n-1)\right] = \sum_{i=1}^{N-1} a_i x(n-i) + \sum_{i=1}^{N-1}\sum_{j=i}^{N-1} b_{ij} x(n-i)x(n-j). \qquad (16)$$

According to Theorem 2, a pth order inverse of (14) is

$$\begin{aligned} z(n) \;=\; & g_p^{-1}\Bigg[u(n) - \sum_{i=1}^{N-1} a_i z(n-i) + \\ & \sum_{i=1}^{N-1}\sum_{j=i}^{N-1} b_{ij} z(n-i)z(n-j)\Bigg]. \end{aligned} \qquad (17)$$

The pth order inverse $g_p^{-1}[\cdot]$ can be computed iteratively as in [6] and is given by

$$g_p^{-1}\left[u(n)\right] = -g_1^{-1}\left[q_p\left[g_{p-1}^{-1}\left[u(n)\right]\right] - u(n)\right], \qquad (18)$$

where $g_1^{-1}[\cdot]$ is the inverse of the first Volterra operator of $g[\cdot]$ (*i.e.*, a_0^{-1} in our case) and $q_p[\cdot]$ is the truncated Volterra series expansion of the system $g[\cdot]$ that contains only the second through pth order Volterra kernels.

The computational cost expressed in multiplications for the evaluation of (17) is $2(N-1) + \frac{(N-1)N}{2} + (N+2)(p-1)$. The corresponding computational cost for directly computing the pth order inverse of (14) as in [6] is $N + \left(2N + \frac{N(N+1)}{2}\right)(p-1)$. If the order p is greater than two the computational advantage of using (17) becomes evident. Implementing (17) has computational cost of $O(N^2 + pN)$ multiplications while the method in [6] requires $O(N^2 p)$ multiplications. In general, if we want to derive a pth order inverse for a Volterra filter of order q, the methodology suggested by Theorem 2 is more convenient when p is greater than q. On the other hand, when $p < q$ only the first p Volterra operators are significant for the evaluation of the pth order inverse. In this situation, both methods of inversion require almost the same number of multiplications for computing each output sample.

4. AN EXPERIMENTAL RESULT

We consider the pth order inversion of the second order Volterra filter with input-output relationship

$$\begin{aligned} y(n) \;=\; & x(n) - x(n-1) - 0.125x(n-2) + \\ & 0.3125x(n-3) + x^2(n) - 0.3x(n)x(n-1) + \\ & 0.2x(n)x(n-2) - 0.5x(n)x(n-3) + \\ & 0.5x^2(n-1) - 0.3x(n-1)x(n-2) + \\ & -0.6x(n-1)x(n-3) - 0.6x^2(n-2) + \\ & 0.5x(n-2)x(n-3) - 0.1x^2(n-3). \end{aligned} \qquad (19)$$

The pth order inverse derived applying Theorem 2, where $g_p^{-1}[\cdot]$ is computed as in [6], is compared with the pth order inverse obtained by directly using the method in [6]. In Figure 1 the points identified with o refer to the pth order inverse of the Theorem 2, while the points indicated with + refer to the pth order inverse of [6]. The plots in Figure 1a compare the computational cost in multiplications for different orders p of the inversion. The computational efficiency of the pth order inverse of Theorem 2 over the inverse suggested in [6] can be clearly seen in this figure. Figures 1b and 1c displays the mean-squared error (MSE) between the input signal of the system in (19) and the output of its pth order inverse when connected in cascade to the system. The input signal was white and Gaussian-distributed with zero mean value. Figure 1b presents the MSE in the reconstruction of the input for different values of the inverse filter order p when the standard deviation of the input signal was 0.05. Figure 1c shows the mean-square error values for different standard deviations of the input signal for a fifth-order inverse system. All the results presented are time averages of 1,000 sam-

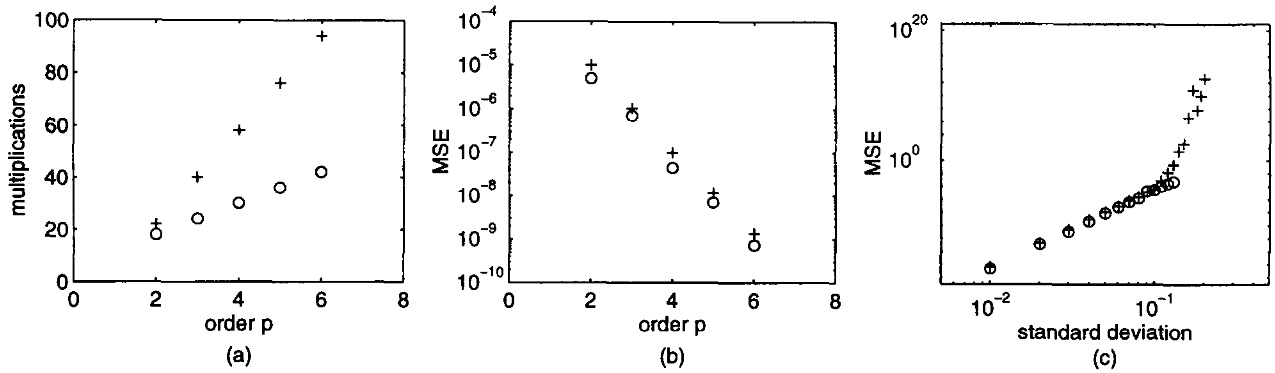

Figure 1: Experimental Results.

ples of the ensemble averages computed over fifty independent experiments. Values of the standard deviations for which a corresponding MSE value is absent correspond to instability situations. We can see that our approach give the similar or better performances as the method in [6] till instability arises in the inverse system. In such situations, the performance of the pth order inverse of [6] are also unacceptable.

5. CONCLUDING REMARKS

This paper presented two theorems for the exact inverse and the pth order inverse of a wide class of discrete-time nonlinear systems. As in the linear case, even if a nonlinear system is BIBO stable its inverse system may be unstable. The inverse systems we consider in this paper are in most cases recursive nonlinear filters and therefore may possess poor stability properties. Consequently, the stability of such systems must be tested after the inversion of the filter. Stability of recursive nonlinear systems is still a topic of active research. Some useful results for the stability of recursive polynomial filters can be found in [1, 2, 3, 4, 5, 9].

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