BINARY POLYNOMIAL TRANSFORMS FOR NONLINEAR SIGNAL PROCESSING

Jaakko Astola, Karen Egiazarian, Ruşen Öktem

Signal Processing Laboratory
Tampere University of Technology
Tampere, FINLAND

Sos Agaian

Electrical Engineering Department
University of Texas at San Antonio
San Antonio, Texas, USA

ABSTRACT

In this paper we introduce parametric binary Rademacher functions of two types. Based on them using different kinds of arithmetical and logical operations we generate a set of binary polynomial transforms (BPT). Some applications of BPT in nonlinear filtering and in compression of binary images are presented.

1. INTRODUCTION

Spectral analysis is a powerful tool in communication, signal/ image processing and applied mathematics and there are many reasons behind the usefulness of spectral analysis. First, in many applications it is convenient to transform a problem into another, easier problem; the spectral domain is essentially a steady-state viewpoint; this not only gives insight for analysis but also for design. Second, and more important: the spectral approach allows us to treat entire classes of signals which have similar properties in the spectral domain; it has excellent properties such as fast algorithms, high energy compaction property of the transform data.

However, there are some problems in signal/ image processing (especially if the input data is binary) where they cannot be applied directly.

We may properly ask what is a basic transform for binary signal/image processing which plays the same role for it as spectral transform for linear processing.

Similarly we may ask what is a basic transform for nonlinear signal/image processing which plays the same role for it as spectral transform for linear processing.

We will develop a compact representation of signals for later use in digital logic and nonlinear signal analysis. Our approach will be based on introduction of parametric binary Rademacher functions (BRF); on generating a set of transforms using Rademacher functions and different kinds of arithmetical and logical operations. Emphasis is placed on the two classes of transforms, introduced in this paper: transforms generated by BRFs using only one operation (arithmetical or logical) and transforms generated by BRFs using two operations (arithmetical or logical). We present the basic properties of these transforms.

This development is motivated also by potential applications in areas such as nonlinear signal/image processing, construction of new architectures of signal/image processing computers or systems suitable for simultaneous arithmetical or logical operations.

2. BINARY POLYNOMIAL FUNCTIONS AND MATRICES

2.1. Rademacher Functions and Matrices

Let a and b be arbitrary integers. We form the classes $r_n(t,a,b)$ and $s_n(t,a,b)$ of real periodic functions in the following way. Let

$$r_0(t,a,b) \equiv s_0(t,a,b) \equiv 1, \tag{1}$$

$$r_{n+1}(t,a,b) = \begin{cases} a, & \text{for } t \in \bigcup_{m=0}^{2^n-1} [\frac{m}{2^n}, \frac{m}{2^n} + \frac{1}{2^{n+1}}), \\ b, & \text{for } t \in \bigcup_{m=0}^{2^n-1} [\frac{m}{2^n} + \frac{1}{2^{n+1}}, \frac{m+1}{2^n}). \end{cases} \tag{2}$$

and

$$s_{n+1}(t,a,b) = \begin{cases} a, & \text{for } t \in [0, \frac{1}{2^{n+1}}), \\ b, & \text{for } t \in [\frac{1}{2^{n+1}}, \frac{1}{2^n}) \\ 0, & \text{otherwise for } t \in [0,1), \end{cases} \qquad n = 0,1,2,\ldots \tag{3}$$

Note that

$$r_1(t,a,b) = s_1(t,a,b) = \begin{cases} a, & \text{for } t \in [0, \frac{1}{2}), \\ b, & \text{for } t \in [\frac{1}{2}, 1) \end{cases} .$$

DEFINITION 2.1: The functions $r_n(t,a,b)$ are called *Rademacher* (a,b)*-functions of I-type.* The functions $s_n(t,a,b)$ are called *Rademacher* (a,b)*-functions of II-type.*

Note that when $a = 1$ and $b = -1$, the Rademacher (a,b)-functions of I-type coincide with the Rademacher functions [5]. Let us consider some properties of the generalized Rademacher functions defined above.

PROPERTY 2.1: There are the following representations of the Rademacher (a,b)-functions of the I-type:

$$1)\ r_{n+1}(t,a,b) = a + (b-a)c_{n+1}(t),\ t \in [0,1), \tag{4}$$

where c_{n+1} are the coefficients from

$$t = \sum_{n=1}^{\infty} c_n(t) 2^{-n} = \sum_{n=1}^{\infty} c_n 2^{-n}. \tag{5}$$

2) $r_n(t,a,b) = \frac{1}{2}\Big[(a-b)\text{sgn}\,[\sin(2^n\pi t)] + a + b\Big]$,

$$t \neq \frac{k}{2^n}, k = 0, ..., 2^{n-1},\ \text{sgn}\,(y) = \begin{cases} 1, & \text{if } y > 0, \\ -1, & \text{if } y < 0. \end{cases}$$

3) There is the following representation of the Rademacher functions of II-type:

$$s_{n+1}(t,a,b) = [a + (b-a)c_{n+1}(t)]\prod_{i=1}^{n} c_i(t). \qquad (6)$$

4) There is the following relation between the Rademacher (a,b)-functions of I-type and II-type:

$$r_{n+1}(t,a,b) = \sum_{m=0}^{2^{n-1}-1} s_{n+1}(t - \frac{m}{2^{n-1}}, a, b). \qquad (7)$$

For a natural number n and integers a and b we define the following discrete functions:

$$R_k^{(n)}(x,a,b) = r_k(\frac{2x+1}{2^{n+1}}, a, b), \qquad (8)$$

$$k = 0, 1, ..., n-1;\ x = 0, 1, ..., 2^n - 1;\ n \geq 0.$$

$$S_k^{(n)}(x,a,b) = s_k(\frac{2x+1}{2^{n+1}}, a, b), \qquad (9)$$

$$k = 0, 1, ..., n-1;\ x = 0, 1, ..., 2^n - 1;\ n \geq 0.$$

DEFINITION 2.2: The rectangular $(2^n \times (n+1))$ matrix $\mathbf{R}(n,a,b)$, whose $(x,k)^{\text{th}}$ element is $R_k^{(n)}(x,a,b)$, $x = 0, 1, \cdots, 2^n - 1$, $k = 0, \cdots, n$, is called *Rademacher* (a,b) *matrix of I-type* (of order n). The rectangular $(2^n \times (n+1))$ matrix $\mathbf{S}(n,a,b)$, whose $(x,k)^{\text{th}}$ element is $S_k^{(n)}(x,a,b)$, $x = 0, 1, \cdots, 2^n - 1$, $k = 0, \cdots, n$, is called *the Rademacher* (a,b) *matrix of II-type* (of order n).

REMARK 2.1: For each pair of integers (n,k), $n > 0$, $1 \leq k \leq n$, the k^{th} column of $\mathbf{R}(n,a,b)$ defines a function $\{0, 1, \cdots, 2^n - 1\} \to \{a, b\}$. This set of functions is called the system of *discrete Rademacher* (a,b) *functions of I-type*. Analogously, for each pair of integers (n,k), $n > 0$, $1 \leq k \leq n$, the k^{th} column of $\mathbf{S}(n,a,b)$ defines a function $\{0, 1, \cdots, 2^n - 1\} \to \{a, b\}$, the system of which is called the system of *discrete Rademacher* (a,b) *functions of II-type*.

EXAMPLE 2.1: When $n = 4$, the Rademacher (a,b)-matrices of I-type $\mathbf{R}^T(4,a,b)$ and II-type $\mathbf{S}^T(4,a,b)$ have the following forms, respectively:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ a & a & a & a & a & a & a & a & b & b & b & b & b & b & b & b \\ a & a & a & a & b & b & b & b & a & a & a & a & b & b & b & b \\ a & a & b & b & a & a & b & b & a & a & b & b & a & a & b & b \\ a & b & a & b & a & b & a & b & a & b & a & b & a & b & a & b \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ a & a & a & a & a & a & a & a & b & b & b & b & b & b & b & b \\ a & a & a & a & b & b & b & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & a & b & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where T denotes matrix transposition.

The discrete Rademacher (a,b)-matrices of I-type and II-type have the following properties.

PROPERTY 2.2:

1) $R_k^{(n)}(x,a,b) = a + (b-a)\cdot x_k(x);\ k = 1, ..., n$,

where n is a natural number, and x, $0 \leq x \leq 2^n - 1$ is written in the form

$$x = \sum_{j=0}^{n-1} x_j(x)\cdot 2^{n-j-1},\ j = 0, 1, ..., n-1. \qquad (10)$$

2) $\mathbf{R}^T(n,a,b)\cdot\mathbf{R}(n,a,b) = 2^{n-2}[(a-b)^2\cdot\mathbf{I}_{(n)} + (a+b)^2\cdot\mathbf{J}_{(n)}]$,
where $\mathbf{I}_{(n)}$ is the identity matrix of order n and $\mathbf{J}_{(n)}$ is the $(n \times n)$ matrix whose all entries are equal to 1.

In particular, for the classic Rademacher matrices

$$\mathbf{R}^T(n,1,-1)\cdot\mathbf{R}(n,1,-1) = 2^n\cdot\mathbf{I}_{(n)}.$$

3) Let n be a natural number, $1 \leq k \leq n$ and suppose that x, $0 \leq x \leq 2^n - 1$ is written in the form (10). Then

$$S_k^{(n)}(x,a,b) = [a + (b-a)\cdot x_k(x)]\prod_{i=0}^{k-1}[1 - x_i(x)];\ k = 1, ..., n. \qquad (11)$$

4) $\mathbf{S}^T(n,a,b)\cdot\mathbf{S}(n,a,b) = a(a+b)\cdot[\mathbf{J}_{(n)} + \sum_{i=1}^{n-2} 2^i\mathbf{P}_{(n)}^{(i)}]$ $+ b(b-a)\cdot\mathbf{D}_{(n)}$,
where $\mathbf{J}_{(n)}$ is the $(n \times n)$ matrix whose all the entries are equal to 1, $\mathbf{P}_{(n)}^{(i)}$ is the $(n \times n)$ matrix whose leftmost $(n-i) \times (n-i)$ submatrix is $\mathbf{J}_{(n-i)}$, $i = 1, ..., n-2$, and

$$\mathbf{D}_{(n)} = \text{diag}(2^{n-1}, 2^{n-2}, ..., 2, 1).$$

2.2. $(a,b;\tau)$-Polynomial Functions of I-Type

Let τ be an arbitrary associative arithmetical or logical operation, $(m_0, m_1, ..., m_{n-1})$ be the binary representation of an integer $m = \sum_{i=0}^{n-1} m_i 2^{n-1-i}$, $0 \leq m \leq 2^n - 1$. Let also $\mathbf{H}\otimes\mathbf{G}$ be the Kronecker product of the matrix $\mathbf{H}$ and the matrix $\mathbf{G}$, and $\mathbf{H}^{\otimes n}$ be the Kronecker n^{th} power of the matrix $\mathbf{H}$, i.e. $\mathbf{H}^{\otimes n} = \underbrace{\mathbf{H}\otimes\mathbf{H}\otimes ...\otimes\mathbf{H}}_{n \text{ times}}$.

We form a class of functions $\phi_m^{(n)}(x,a,b;\tau)$ from the system of discrete Rademacher (a,b)-functions of I type (i.e. from $\{R_m^{(n)}(x,a,b)\}_{m=0}^{n-1}$) in the following way. Let

$$\phi_m^{(n)}(x,a,b;\tau) = \phi_{m_0,m_1,...,m_{n-1}}^{(n)}(x,a,b;\tau)$$

$$= [R_0^{(n)}(x,a,b)]^{m_0}\ \tau\cdots\ \tau\ [R_{n-1}^{(n)}(x,a,b)]^{m_{n-1}}, \qquad (12)$$

where τ is an arithmetical or logical operation. If τ is a logical operation, then we naturally assume $a, b \in \{0, 1\}$.

DEFINITION 2.3: The functions $\phi_m^{(n)}(x,a,b;\tau)$, $m = 0, ..., n-1$, are called $(a,b;\tau)$-*polynomial functions*. If τ is an arithmetical (logical) operation, the $(a,b;\tau)$-polynomial functions are called arithmetical (logical).

DEFINITION 2.4: The square matrix $\Phi(n, a, b; \tau) = [\phi_m^{(n)}(x, a, b; \tau)]$, $x, m = 0, 1, ..., 2^n - 1$, which has as columns the $(a, b; \tau)$-polynomial functions, is called *the* $(a, b; \tau)$*- polynomial matrix.*

Let τ be multiplication operation. The $(a, b, \times)$-polynomial functions $\phi_m^{(n)}$ are formed via Rademacher (a, b)-functions $R_m^{(n)}(x, a, b; \times)$ by the following formula:

$$\phi_m^{(n)}(x, a, b; \times) = \prod_{i=0}^{n-1} [R_i^{(n)}(x, a, b)]^{m_i},$$

where m_i is the i^{th} bit in the binary representation of m via $m = \sum_{i=0}^{n-1} m_i 2^{n-1-i}$.

When $a = 1, b = -1$, the (a, b)-polynomial functions coincide with the Walsh functions, and when $a = 0, b = 1$, they coincide with the Reed-Muller (conjunctive) functions [1].

Based on Property 2.2, 1) we have the following expressions:

$$\phi_m^{(n)}(x, 1, -1) = \prod_{i=0}^{n-1} (1 - 2x_i)^{m_i}, \tag{13}$$

for the Walsh functions, and

$$\phi_m^{(n)}(x, 0, 1) = \prod_{i=0}^{n-1} x_i^{m_i}, \tag{14}$$

for the conjunctive functions.

Let now τ be an arbitrary associative binary logical operation, $a = 0$ and $b = 1$. Let m and x have binary representations $(m_0, \cdots, m_{n-1})$ and $(x_0, \cdots, x_{n-1})$ respectively. Then, by Property 2.2, the formula (12) becomes

$$\phi_m^{(n)}(x, 0, 1; \tau) = \phi_{m_0, ..., m_{n-1}}^{(n)}(x, 0, 1; \tau) = \tau_{i=0}^{n-1} x_i^{m_i}.$$

Examples of the τ-polynomial logical functions are:

a) conjunctive, if τ is the operation $\wedge$ (AND); denoted by $\phi_m^{(n)}(x, \wedge)$;

b) disjunctive, if τ is the operation $\vee$ (OR); denoted by $\phi_m^{(n)}(x, \vee)$;

c) antivalent, if τ is the operation $\oplus$ (XOR); denoted by $\phi_m^{(n)}(x, \oplus)$.

2.3. (a, b)-Polynomial Functions of II-type

First define the shifted Rademacher (a, b)-functions of II-type, namely

$$s_m^{(k)}(x, a, b) = s^{(k)}(x - \frac{m}{2^k}, a, b), \tag{15}$$

where $m \in \{0, 1, ..., 2^k - 1\}$.

DEFINITION 2.5: The functions $h_0^{(n)}(x, a, b) = s^{(0)}(x, a, b)$, $h_1^{(n)}(x, a, b) = s^{(1)}(x, a, b)$, and $h_{2^k+m}^{(n)}(x, a, b) = s_m^{(k+1)}(x, a, b)$, $m = 0, 1, ..., 2^k - 1$, $k = 1, ..., n - 1$, are called (a, b)*-polynomial functions of II-type*, and they form *the system of* (a, b)*-polynomial functions of II-type.*

DEFINITION 2.6: The square matrix $\mathbf{H}(n, a, b) = [h_r^{(n)}(x, a, b)]$, $x, r = 0, 1, ..., 2^n - 1$, which has as columns the (a, b)-polynomial functions of II-type, is called *the* (a, b)*-polynomial matrix of II-type.*

Note that the classical Haar functions (matrices) are the particular case of the (a, b)-polynomial functions (matrices) of II-type where $a = 1, b = -1$. The Haar-conjunctive functions (matrices) will be $(0, 1)$-polynomial functions (matrices) of II type.

EXAMPLE 2.2: The Haar-conjunctive $\mathbf{H}_3$ matrix is:

$$\mathbf{H}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

3. BINARY POLYNOMIAL TRANSFORMS

3.1. (a, b)-Polynomial Transforms of II-type as Binary Wavelet Transforms

Binary Wavelet Transform emerged from the application of wavelet theory in finite field with two elements $\mathbf{GF}(2)$ [4, 7]. It is highly advantageous in computational point of view, since the computations are performed by "exclusive or" and "and" operations. Although smoothness and vanishing moments properties of real discrete wavelet transforms [9] do not make sense in $\mathbf{GF}(2)$, BWT is useful to localize data in time and frequency and separately encode rapid and slow changes across the data. In that sense, an example for a one stage BWT is defined as

$$BWT = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & \mathbf{I} \end{pmatrix} \mathbf{L} \tag{16}$$

in [4], where $\mathbf{L}$ is a permutation matrix called "lazy wavelet transform" such that $L(x_1, x_2, \ldots) = [x_2, x_4, \ldots, x_1, x_3, \ldots]$. Wavelet vectors resulting from (16) are exactly the same as the columns of Haar-conjunctive matrix ((0, 1)-Polynomial Matrix of II-type) introduced in the previous Section.

3.2. (a, b)-Polynomial Functions of I-type as Discrete Wavelet Packet Transforms

Wavelet packet transform is a generalization of wavelet transform which offers a wider range of analysis possibilities for the signal [9]. It is associated with a best basis selection algorithm which selects a subdecomposition structure among all possible decomposition structures presented by the packet transform, subject to a criterion. Wavelet Packet Transform can be better visualized by a full binary tree, where left and right branchings represent lowpass and highpass filterings, respectively, and best basis selection corresponds to extracting a subtree of the binary tree.

Consider the Walsh matrix $\mathbf{W}_n$ of order n in Hadamard ordering. It corresponds to a Wavelet Packet Transform with [1 1] and [1, −1] being the lowpass and highpass filters, respectively. The conjunctive matrix $\mathbf{K}_n$ of order n is also a wavelet packet transform in $\mathbf{GF}(2)$ with [1 0] and [1 1] acting as the lowpass and highpass filters, respectively.

3.3. Efficient computation algorithms

3.3.1. Fast (0,1)-Rademacher Transforms of I-type

LEMMA 3.1: The $(0,1)$-Rademacher matrix $\mathbf{R}_n = \mathbf{R}(n,0,1)$ of I-type can be represented as

$$\mathbf{R}_n = \prod_{j=0}^{n-1} \mathbf{R}_n^{(j)}, \tag{17}$$

where

$$\mathbf{R}_n^{(0)} = \begin{pmatrix} \mathbf{I}_{n-1} & \mathbf{0}_{n-1} \\ \mathbf{I}_{n-1} & \mathbf{1}_{n-1} \end{pmatrix}, \tag{18}$$

$$\mathbf{R}_n^{(j)} = \begin{pmatrix} \mathbf{I}_{n-1-j} & \mathbf{0}_{n-1} & 0 \\ \mathbf{I}_{n-1-j} & \mathbf{1}_{n-1} & 0 \\ 0 & 0 & \mathbf{I}_{(j)} \end{pmatrix},\ j = 1,2,...,n-2, \tag{19}$$

$$\mathbf{R}_n^{(n-1)} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & & & \\ 0 & & & \\ \vdots & & \bar{\mathbf{I}}_{(n)} & \\ 0 & & & \end{pmatrix}, \tag{20}$$

$\mathbf{I}_j$ (respectively, $\mathbf{I}_{(j)}$) is the identity matrix of order 2^j (respectively, of order j), $\bar{\mathbf{I}}_{(n)}$ is an opposite identity matrix of order n, $\mathbf{0}_j$ (respectively, $\mathbf{1}_j$) are column-vectors of length 2^j, consisting of zeros (respectively, of ones).

3.3.2. Fast $(a,b,\times)$-Polynomial Transforms of I-Type

LEMMA 3.2: The $(a,b,\times)$-polynomial matrices $\mathbf{\Phi}_n = \mathbf{\Phi}(n,a,b;\times)$ can be expressed by the following formula:

$$\mathbf{\Phi}_n = (\mathbf{G}_n)^n, \tag{21}$$

where

$$\mathbf{G}_n = \begin{pmatrix} 1 & a & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & a & \dots & 0 & 0 \\ \dots & & \dots & & \dots & & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & a \\ 1 & b & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & b & \dots & 0 & 0 \\ \dots & & \dots & & \dots & & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & b \end{pmatrix} \tag{22}$$

3.3.3. Fast (a,b)-Polynomial Transforms of II-Type

LEMMA 3.3: The (a,b)-polynomial matrices of II-type $\mathbf{H}_n = \mathbf{H}(n,a,b)$ can be expressed by the following formula:

$$\mathbf{H}_n = \mathbf{G}_n \prod_{j=1}^{n-1} diag(\mathbf{G}_{n-j}, \mathbf{I}_j, ..., \mathbf{I}_{n-1}), \tag{23}$$

where

$$\mathbf{G}_k = [\mathbf{I}_{k-1} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ \ \mathbf{I}_{k-1} \otimes \begin{pmatrix} a \\ b \end{pmatrix}], \tag{24}$$

where

$$\mathbf{H}_1 = \mathbf{G}_1 = \begin{pmatrix} 1 & a \\ 1 & b \end{pmatrix}$$

and $\mathbf{I}_r$ is the identity matrix of order 2^r.

4. BINARY POLYNOMIAL TRANSFORMS IN NONLINEAR FILTERING

The use of BPT in nonlinear filtering was discussed in details in [1]. Here we just show very simple and very efficient realization for a threshold filter [3]. The threshold filter (TF) $S_f(X)$ based on the M-variable discrete function $f(\cdot) : \{0,1\}^M \to \Re$ maps the signal X into the output signal $Y = \{Y(k), k = 1,2,...,L\}$, where

$$Y(k) = S_f(\mathbf{X}(k)) = \sum_{n=1}^{R} f(\sigma_n(\mathbf{X}(k))), \tag{25}$$

$\sigma_n(.)$ is the vectoral threshold function with the elements defined by $\sigma_n(X) = \begin{cases} 1 & \text{if } X \geq n \\ 0 & \text{otherwise.} \end{cases}$

Denote $\mathbf{x}_j = \sigma_j(\mathbf{X})$), $j = 1,...,M$. Define an indicator

$$\chi(\mathbf{x}_j) = [\chi_0(\mathbf{x}_j),...,\chi_{2^N-1}(\mathbf{x}_j)];\ \ j = 1,...,M, \tag{26}$$

where

$$\chi_k(\mathbf{x}_j) = \begin{cases} 1, & \text{if } (k_{N-1},...,k_0) = \mathbf{x}_j, \\ 0, & \text{otherwise,} \end{cases} \tag{27}$$

$j = 1,...,M$; $(k_{N-1},...,k_0)$ is the binary vector representation of the positive integer k, $k = 0,...,2^N - 1$.

Denote by h the histogram vector of $\mathbf{x}_j$, $j = 1,...,M$, i.e.

$$\mathbf{h} = \sum_{j=1}^{M} \chi(\mathbf{x}_j). \tag{28}$$

The matrix analogue of the formula (25) is the following (index n is omitted)

$$y = \mathbf{f} \cdot \mathbf{h}^T, \tag{29}$$

where $\mathbf{h}^T$ is the transposed vector of the histogram vector $\mathbf{h}$ (28). Let us now modify (29) to the form

$$y = \mathbf{h} \cdot \mathbf{H} \cdot \mathbf{g}^T, \tag{30}$$

with

$$\mathbf{g}^T = \mathbf{H}^{-1} \cdot \mathbf{f}, \tag{31}$$

where $\mathbf{H}$ is any nonsingular matrix. In fact, realization of threshold filter in [3] follows directly from (30)-(31) when $\mathbf{H} = \mathbf{K}$ is the conjunctive (Reed-Muller) matrix. Nevertheless, there exists much simpler and efficient realization of this filter using Rademacher-sorting network without computation of the threshold histogram $\mathbf{h}$. The flowchart structure of Rademacher-sorting network is similar to the flowchart of the fast Rademacher transform, with changing of addition operations to minimum operations.

5. BINARY POLYNOMIAL TRANSFORMS IN COMPRESSION OF BINARY IMAGES

Binary image compression may find applications in compressing bi-level high resolution images such as fax pages, scanned images or segmentation data and bit-plane by bit-plane compression of multi-level images in some specific cases. Recently, multiresolution analysis via wavelet transform has found efficient applications in multilevel image compression ([9]). The success of multilevel image compression with wavelets has been followed by applications of wavelet theory over the finite field $\mathbf{GF}(2)$ ([4, 7]). Swanson and Tewfik developed a theory of binary wavelet transform in terms of two band perfect reconstruction filter banks in ([7]). Their test results with binary image compression achieved promising results in terms of first order entropy. Soon after Johnston ([4]) used another approach (lifting scheme ([8])) to construct a binary wavelet transform which is a cascade of binary matrices composed of upper and lower unimodular binary blocks and a simple permutation matrix collecting even samples to upper half and odd samples to lower half (see (16)). Here we will present how to obtain similar binary transforms via binary polynomial matrices.

Consider the Reed-Muller matrix $\mathbf{K}_M$ of order M

$$\mathbf{K}_M = K_1^{\otimes M} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{\otimes M} \tag{32}$$

and a binary vector $\mathbf{f}_M$ of length 2^M. The Reed-Muller Transform of $\mathbf{f}_M$ can be defined as

$$\mathbf{f}_{RM} = \mathbf{K}_M \mathbf{f}_M,$$

where the output $\mathbf{f}_{RM}$ is computed in modulo 2 arithmetic. $\mathbf{f}_{RM}$ corresponds to a wavelet packet representation of $\mathbf{f}_M$ which can be obtained by a full wavelet tree decomposition. In this decomposition first and second rows of $\mathbf{K}_1$ act as lowpass and highpass filters, respectively. The above transform can be generalized into two-dimensional case for images as

$$\mathbf{F}_{RM} = \mathbf{K}_M \mathbf{F}^T \mathbf{K}_M^T, \tag{33}$$

where $\mathbf{F}$ is a binary matrix representing pixel values of a binary image. Note that (33) is an invertible transform where the inverse of $\mathbf{K}_M$ in $\mathbf{GF}(2)$ is equal to itself. We can obtain the best multiresolution decomposition of $\mathbf{F}$, by selecting a sub-decomposition structure covered by $\mathbf{K}_M$. The subdecomposition structure can be obtained by forming a matrix such that

$$\mathbf{K}'_M = \begin{bmatrix} \mathbf{A} & \mathbf{0}_{M-1} \\ \mathbf{B} & \mathbf{C} \end{bmatrix}, \tag{34}$$

where $\mathbf{0}_{M-1}$ is the zero matrix with size $2^{M-1} \times 2^{M-1}$, and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ can be either $\mathbf{0}_{M-1}$ or $\mathbf{K}'_{M-1}$ with the restriction that

$$\mathbf{K}'_1 = \mathbf{K}_1.$$

Some quantitative results of the above binary transform are presented in Table 1. For all four test images, the logarithmically growing decomposition tree is found to give the best performance in terms of minimum number of zeros. Hence, the only overhead information needed is the depth of the tree. For comparison, the results of S and S&P Transform [6] are also included. Although S and S&P are real wavelet transforms for lossy and lossless compression, they is also suitable for binary image compression because they produce integer outputs in a very limited range for binary data. The proposed binary transform performs better than both S and S&P in terms of first order entropy of transformed coefficients. It satisfies significant reduction of entropy from the original image.

Table 1: Compression results for thresholded 256×256 Lena, 128×128 Ball, 128×128 Bird and 512×512 Text images

	org. Lenna	S&P	S	BT
no.nonzero	26924	3940	3688	4251
Entropy	0.98	0.39	0.37	0.34
	org.Bird	S	S&P	BT
no.nonzero	12940	843	747	730
Entropy	0.74	0.35	0.32	0.26
	org. Ball	S&P	S	BT
no.nonzero	56715	1727	1599	1531
Entropy	0.75	0.06	0.06	0.05
	org. Text	S	S&P	BT
no.nonzero	3174	366	318	399
Entropy	0.71	0.19	0.17	0.16

6. REFERENCES

[1] Agaian, S.S., Astola, J., and Egiazarian, K. (1995) *Binary Polynomial Transforms and Nonlinear Digital Filters*, Marcel Dekker Inc.

[2] Beauchamp, K. G. (1975). *Walsh Functions and Their Applications*, Academic Press.

[3] Jeong, H., Song, J., and Lee, Y. H. (1995). Extended threshold Boolean filters: representation and design, *Proc. of 1995 IEEE Workshop on Nonlinear signal processing*, ed. I.Pitas, vol. II, pp. 911-914.

[4] Johnston, C. P. (1996). The Lifting Scheme and Finite Precision Error Free Filter Banks, *SPIE* vol. 2825, pp. 307-316.

[5] Rademacher, H. (1922). Einige Satze von Allgemainen Orthogonal Funktionen, *Math. Annalen*, 87, pp. 122-138.

[6] Said A. and Pearlman, W.A. (1996). An Image Multiresolution Representation for Lossless and Lossy Compression, *IEEE Transactions on Image Processing*, vol. 5, no. 9, pp. 1303-1310.

[7] Swanson, M.D. and Tewfik, A.H. (1996). A Binary Wavelet Decomposition of Binary Images, *IEEE Transactions on Image Processing*, vol. 5, no. 12.

[8] Sweldens, W. (1994). The Lifting Scheme: A Custom-Design Construction of Biorthogonal Wavelets, *Technical report IMI* 1994:7, revised version.

[9] Vetterli, M. and Kovačević, J. (1995). *Wavelets and Subband Coding*, Prentice Hall.