Modern Geometry of Image Formation

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Abstract

Image formation is quantised by imposing an image induced connection and computing the associated torsion and curvature in terms of differential or integral invariants. Imposing invariance of the image formation under the classical group of movements slot-machines reading out the torsion and the curvature are shown to locate endpoints and other type of interesting topological objects. Requiring instead invariance under the group of anamorphoses and the group of diffeomorphisms of the image only ridges and ruts can be identified through a non-local topological or integral geometric operation.

1 Introduction

The goal in image analysis and pattern recognition is to find a stable and reproducible description and encoding of an image or better its formation, that is slightly affected by certain sets of transformations. These sets include the group of anamorphoses, the classical groups such as that of Euclidean movements and the transformations caused by noise. In order to achieve stability and reproducibility under these transformations one turns to so-called scale-space theories. A modelling of and a smoothing of the image formation process appears to be in the context of these theories indispensable [1].

Our aim is to demonstrate in section 2 that a grey-valued image can be provided with a non-flat image induced connection. Furthermore, that the torsion and the curvature associated with this connection can be measured in terms of differential or integral invariants. These invariants in their turn can be expressed in ordinary algebraic combinations of the image grey-values. It is shown that so-called ridges and ruts of an image are the essential physical objects being invariant under the group of anamorphoses and diffeomorphisms of the image domain. Furthermore, that they can be detected by a non-local topological or integral geometric operation. Section 3 concludes with a discussion of further research.

2 Image Formation

Modern geometry of image formation is treated in subsection 2.1 within the context of differential geometry and in subsection 2.2 within that of integral geometry. It's demonstrated that torsion and curvature of the image formation can be expressed in terms of the image structure and they reveal the topological interesting objects in images such as ridges, ruts and endpoints. For proofs of theorems and more extensive expositions of modern geometry see the references in [1]

2.1 Differential Geometry

Let M be a D-dimensional image domain parametrised by canonical coordinates $p = (p^1, \dots p^D)$. Now consider the frame bundle $F \equiv P(M, \pi, A(D, \mathbb{R}))$ where P is the total space consisting of all frames Φ_p at each point $p \in M$, $\pi : P \to M$ is the projection and $A(D, \mathbb{R}) = GL(D, \mathbb{R}) \rhd T(D, \mathbb{R})$ the full affine group, where $Gl(D, \mathbb{R})$ is the general linear group and $T(D, \mathbb{R})$ the translational group. In this context let's define a local frame as follows.

Definition 1 *A local frame Φ_p is defined by:*

$$\Phi_p = (x; e_1, \dots, e_D)\,(p),$$

where the vectors $(x, e_1, \dots e_D)(p)$ span the local tangent space $T_pA(D, \mathbb{R})$.

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Now an affine connection Γ in the frame bundle F is defined as follows.

Definition 2 *An affine connection Γ in the frame bundle F is defined in terms of the Lie algebra $\mathcal{G}(D,\mathbb{R})$-valued connection one-forms (ω^i, ω_i^j) and the frame vectors $(x, e_1, \ldots e_D)$ through the following equality:*

$$\nabla x = \omega^i e_i, \;\; \nabla e_i = \omega_i^j e_j,$$

where ∇ is the covariant differential operator.

The affine connection Γ satisfies so-called structure equations:

Theorem 1 *Given an affine connection Γ in the frame bundle F, defined in (2), then the connection one-forms satisfy the following structure equations:*

$$\begin{aligned} D\omega^i &= d\omega^i + \omega_k^i \wedge \omega^k = \Omega^i, \\ D\omega_j^i &= d\omega_j^i + \omega_k^i \wedge \omega_j^k = \Omega_j^i, \end{aligned}$$

where d the ordinary exterior derivative, $\wedge$ is the wedge product, D the covariant derivative, Ω^i is the torsion two-form and Ω_j^i is the curvature two-form.

In turn the torsion and the curvature two-form satisfy so-called Bianchi identities:

Theorem 2 *Let Γ be an affine connection in the frame bundle F with torsion two-form Ω_0^i and curvature two-form Ω_j^i. The integrability conditions for the structure equations, that are the Bianchi identities, are given by:*

$$D\Omega^i = \Omega_j^i \wedge \omega^j, \;\; D\Omega_j^i = 0,$$

After this concise summary let us quantise image formation of a grey-valued image L_0 on a two-dimensional Euclidean image domain E^2 onto $\mathbb{R}^+$ in terms of its differential geometry. For generalisations to higher dimensional spatio-temporal grey-valued images possibly endowed with other local geometries the reader is referred to [1]. The essential correspondence between the differential geometries of images on two-dimensional and higher dimensional domains, however, will be indicated.

Definition 3 *A grey-valued image L_0 on a two-dimensional Euclidean space E^2 onto $\mathbb{R}^+$ is defined by a scalar-valued density function:*

$$L_0 : E^2 \to \mathbb{R}^+.$$

Thus the grey-value of a pixel is normally just some weighted integration of this density function over its corresponding spatial neighbourhood. So an image is some tested measure or better distribution.

Besides that we study the meaning of those differential geometric invariants that are the same under the spatially homogeneous Euclidean group action $E(2)$ we also contemplate the essence of those differential geometric invariants not affected by the group of anamorphoses and by the group of (volume preserving) diffeomorphisms of the grey-valued image.

Definition 4 *The group of anamorphoses of grey-valued image (3) is the set of non-constant spatially homogeneous and monotonic grey-value transformations $f : L_0 \to f(L_0)$.*

Note that an anamorphosis may invert the order of the image grey-values. Anamorphoses become important as soon as one would like to find the equivalences in images realised by camera systems with different sensitivity characteristics.

Definition 5 *The group of diffeomorphisms G of the grey-valued image (3) is defined by $g(L_0, x) = (\bar{L}_0, \bar{x})$ with $g \in G$ such that the following conservation law holds:*

$$\int_\Omega L_0 d^2x = \int_{\bar{\Omega}} \bar{L}_0 d^2\bar{x}.$$

This group plays a crucial role as soon as the imaging plane is subjected to a Euclidean movement with respect to the projection center. Level sets of the grey-valued image under this type of transformation do not coincide with central projected versions at all, but the total flux subtended by a fixed spherical angle Θ should of course remain the same unless there occur losses due to the change of the unit normal to the imaging plane with respect to the visual rays [1]. The reason for this non-trivial aspect in the image formation is that a grey-valued image is a density and not a scalar function. In order to perform a sensible analysis despite this group of transformations one "gauges" the vision system in a particular manner, namely by imposing a spherical or elliptic geometry after fixing a distance between the center of projection and the imaging plane [1]. Doing so we can define equivalence relations for images under this group of transformations because they now do not cause any problem as they are nothing more than rotations of the angular coordinatisation of the grey-values. Note that diffeomorphisms of the image by active transformations of the scene can still occur but that they yield only specific equivalences we return to later. Furthermore, that observation space for one view has been simplified considerably, but that the essential equivalence relations and objects consistent with this view do not carry over to other views. The only pragmatic attitude to adopt in dynamic monocular or binocular vision is just to establish dynamic aspect graphs [1] in which each view is a visual event. Last but not least one should realise that an image is the consequence of a sampling at a certain resolution of an intensive physical field. Increasing the the resolution properties yields non-versal deformations (morphisms) of the coarser image. More precisely, an coarse resolution image is a typical recombination of a fine resolution image in which the recombination process is steered by a dynamical scale-space paradigm [1]. The latter morphological changes generated by such a paradigm should not be confused with those caused by an active exploration of the scene that introduce upon approaching an object a relative increase of the inner scale with which this object is observed. As mentioned above these kind of changes should be embedded in a more dynamic scale-space paradigm to prevent ambiguities.

Now let us try to find some interesting image structures invariant under the above defined groups. The essential physical objects of the image L_0 invariant under the first two groups of transformations above are so-called isophotes and flowlines and their Euclidean geometry.

Definition 6 *An isophote C_i of a two-dimensional grey-valued image L_0 is defined by:*

$$L_0(x_1) = c \in \mathbb{R}^+ .$$

Definition 7 *A flowline C_f of a two-dimensional grey-valued image L_0 is defined by:*

$$\frac{dx_2}{dp} = \frac{\partial L_0}{\partial x^i}(x_2(p)); \;\; x_2(0) = x_{20} \in E^2,$$

where $p \in \mathbb{R}$ is an arbitrary parameter.

Let us make explicit some Euclidean differential geometry of the net of isophotes and flowlines by choosing a frame field Φ and a connection (ω^i, ω^i_j) as follows:

$$\begin{aligned} \Phi &= (x_1, x_2; e_1, e_2), \\ (\omega^i, \omega^i_j) &= (ds^i, {\Gamma_{kj}}^i ds^k), \end{aligned}$$

with

$${\Gamma_{kj}}^i = \begin{pmatrix} 0 & (-1)^{k+1}\kappa_k \\ -(-1)^{k+1}\kappa_k & 0 \end{pmatrix}^i_j ,$$

where e_1, s^1, κ_1 and e_2, s^2, κ_2 are unit tangent vector fields, the Euclidean arclengths and curvatures on the isophotes and flowlines, respectively (realise that the latter curvature are just invariant zero-forms being factors in the choice of connection). It is readily shown by means of the Cartan structure equations that the net of isophotes and flowlines with the above connection has both a non-zero torsion tensor $\mathcal{T}$ and a non-zero curvature tensor $\mathcal{R}$:

$$\begin{aligned} \mathcal{T} &= {T_{jk}}^i \omega^j \otimes \omega^k \otimes \epsilon_i, \\ \mathcal{R} &= {R_{ikl}}^j \omega^i \otimes \omega^k \otimes \omega^l \otimes \epsilon_j, \end{aligned}$$

where

$$\begin{aligned} {T_{jk}}^k &= \frac{1}{2}({\Gamma_{jk}}^i - {\Gamma_{kj}}^i), \\ {R_{jkl}}^i &= \frac{d{\Gamma_{jl}}^i}{ds_k} - \frac{d{\Gamma_{kl}}^i}{ds_j} + {\Gamma_{jl}}^m \Gamma^i_{mk} - {\Gamma_{kl}}^n \Gamma^i_{nj}. \end{aligned}$$

Let's continue and find the essential physical objects of the above net invariant under the group of (not necessarily total grey-value preserving) diffeomorphisms of the image domain caused by active transformations. The latter active transformations may lead to anamorphoses of the image or integrable deformations of the net of flowlines and isophotes. It's clear that the set of (non)-isolated singularities and the set of discontinuities of L_0 remain the same topological equivalent sets under these transformations. The vanishing of the image gradient is not affected, neither are discontinuities in L_0. A set of nonisolated singularities occurs, for example, for images like $L_0^n(x,y) = -\Re\Big((x+iy)^{\frac{2n+1}{2}}\Big)$, $n \in \mathbb{N}$. It is not so straightforward to see that this invariance also holds for the landscape of ridges and ruts of the image L_0 [1]. The latter topological equivalence can be explained by the fact that at ridges and ruts the integral curves of the image gradient have opposite convexity. Consequently the connection at the ruts and ridges is completely degenerate implying that any order of derivative with respect to the Euclidean arclength parameter s^1 along the isophotes of the flowline curvature field is vanishing. Because taking all orders into account and the fact that to a finite order there will always be non-ridge or non-rut points for which they are zero, it is impossible to distinguish on the basis of a pure local analysis between ridges, ruts and the borders of their influencing zones consisting of e.g. inflection points. Nevertheless, possible ridges and ruts can be discerned on the basis of a local analysis of the isophote curvature κ_1. If $\kappa_1 > 0$ and $\kappa_2 = 0$, then the points belong to the set of possible ridge points. If $\kappa_1 < 0$ and $\kappa_2 = 0$, then the points belong to the set of possible rut points. In order to find ridges and ruts one has to apply, as will be shown shortly in the next subsection, a non-local topological or integral geometric operation.

The reader might object that these objects are nothing more than some special sets of discontinuities in the second order jet structure and that this multi-jet property not only occurs for singular flowlines. Indeed, the change in convexity is an image induced discontinuity determined by multi-local properties of the exterior derivative field of the image influenced by second order jet information. And for isophotes such changes in convexity do also occur. For example, at the set of non-isolated singularities along the flowlines of the images L_0^n mentioned above the isophotes also change convexity. However, this set can be considered, equally well as the set of extrema of the image, as just parts of the landscape of ridges and ruts. At a n-junction where different components of a set of nonisolated singularities or ridges and ruts are coming together there is just a n-fold branching or a n-fold degeneracy of the image gradient, respectively.

The above image analysis can readily be extended to those on higher dimensional image domains endowed with other local geometries and smoothing schemes [1]. For example in the case of three-dimensional grey-valued images normally the unit normal frame field to an isophote will be unique but for some singular ones they will be multi-valued. Again topological operations and slot-machines for reading out torsion or curvature can be used to locate singularity or discontinuity sets of non-constant co-dimension.

2.2 Integral Geometry

Following Cartan (see [1]) one can apply a displacement to determine the translation vector field and the rotation vector fields to operationalise the torsion and the curvature of the frame bundle F with connection Γ.

Definition 8 *Let Γ be a connection in the frame bundle F. The translation vector field b and the rotation vector fields f_i determined by the connection are defined by:*

$$b = \oint_C \nabla x, \;\; f_i = \oint_C \nabla e_i,$$

where C is an infinitesimally small closed loop and boundary of a two-dimensional submanifold S of M with the same induced connection Γ. The sense of traversing the loop is chosen such that the enclosed submanifold is to the left.

On the basis of the connection one forms ω_0^i a foliation of the manifold (M, Γ) can be realised and choosing $\frac{1}{2}D(D-1)$ pairs of them will yield submanifolds containing the desired submanifold S. These integral invariants are intrinsic vectors of the submanifold (S, Γ) and also of the manifold (M, Γ). Using Stokes' theorem the translation and rotation vector fields can be expressed as [1]:

$$b = \int_S \Omega^i e_i, \;\; f_i = \int_S \Omega_i^j e_j.$$

At vertices of intersecting Volterra surfaces or at singularities where ridges and ruts meet the translation and rotation vector fields satisfy the following superposition principles (conservation laws for "topological" currents as Kirchhoff's law for electric currents):

$$B = \sum b, \;\; F_i = \sum f_i,$$

where the sum is over the different components of the cut lines carrying vector fields b and f_i.

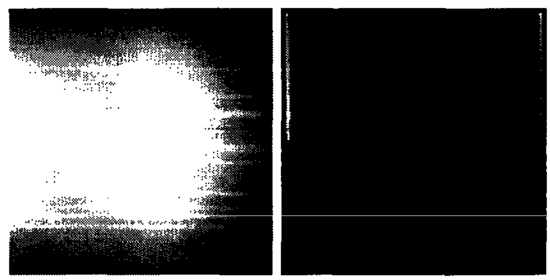

Figure 1: Left frame: a 256×256 pixel-resolution discrete input image $L_0(x, y) = L_0^0(x, y)$. Right frame: the Euclidean length of the translation vector $|b|$ for a linearly scaled version of that image.

Now let us demonstrate that an integral geometric operation suffices to detect certain types of singularity sets. In figure 1 the length of the translation vector field b for a discretised input image L_0 on a two-dimensional Euclidean space E^2 is computed by means of linear scale-space theory [1]. The set of non-isolated singularities will instantaneously disappear upon linear scaling, but the ridges and ruts, and other type of discontinuities can be nicely detected.

Imposing invariance under the group of Euclidean movements and the group of anamorphoses the Euclidean geometry of the net of flowlines and isophotes does matter. Computing the length $|b|$ shows clearly that the isophote curvature κ_1 is high on $(x, y) \in \mathbb{R}^- \times 0$ and that the flowline curvature κ_2 increases on $(x, y) \in \mathbb{R}^- \times 0$ approaching the origin $(x, y) = (0, 0)$. Apparently the translation vector b is a perfect slot-machine to locate cut lines and endpoints that are the essential topological objects in two-dimensional images such as fingerprint images and images of vesseltrees.

Imposing in addition invariance under (volume preserving) diffeomorphisms of the image domain only ridges and ruts can be distinguished on the basis of the valencies of the vertices and the energy values enclosed by them. The detection of ridges and ruts can be realised, as mentioned in the previous subsection, by a non-local topological or integral geometric operation with respect to the unit normal field of the set of flowlines along the isophotes. In the following we'll make these methods explicit and illustrate them.

In order to actually find by a non-local topological operation ridges and ruts parametrise the image by means of e.g. the x^2-axis and trace the extrema of the image gradient length on each line for which x^2-value is constant. Local minima and maxima in the image gradient length on these lines then correspond to rut and ridge points, respectively. Alternatively, take a strip of thinkness of one just noticeable isophote and walk around the global maximum and keep track of the extrema of the length of the image gradient field upon encircling it with the next just noticeable isophote. This supplies us with another non-local topological method for finding ridges and ruts that is equivalent with the well-known watershed method applied in mathematical morphology.

The non-local integral geometric operation consists of a difference operation in a distributional sense (integral invariant manner) at the ridges and ruts along the isophotes with respect to the unit normal field of the sets of flowlines on either side of these singular flowlines. In the neighbourhood of inflections of the flowlines the unit normal field can not be defined but on either side of the isophote passing through the inflections the convexity of the flowlines is the same yielding consequently zero output. Note again performing such an operation on the input image of figure 1 again highlights the x^--axis as cut line and the origin as endpoint. The image formation can be summarised as taking a half-infinite strip on which is defined a ramp image and wrapping this sub-image to and forth the origin over the the x^--axis.

3 Further Research

In this paper modern geometry has been proven to be extremely useful in quantifying the image formation of grey-valued images and localising essential physical objects like ridges, ruts and endpoints. This geometric expertise allows us now to conceive an image as a finite CW-complex in which the ridges, ruts and other type of singularity sets are the essential physical objects bordering different image formation processes. A study of these CW-complexes and particular paths on them, for example paths in three-dimensional images on ribbon knots, can be quite fruitful in establishing the topological aspects involved in the global image formation processes. The latter topological aspects might then for instance be quantified in terms of so-called (generalised) Vassiliev invariants specifying the dynamical processes involved. In this context it might be also interesting to find other topological invariants by applying Chern-Simons perturbation theories. Establishing equivalence relations in terms of these kind of topological invariants might be worthwhile in case of a description of dynamical aspect graphs. Moreover, the "half-space method" introduced in [1] can refine the local multi-jet structure (of the topological currents) considerably and thus the equivalence relations on the CW-complexes.

In [1] the author proposed to smooth the vector density fields quantising the image formation in a non-linear differential or integral geometric manner. In this context it might be interesting to formulate also so-called dynamic scale-space theories that are topologically equivalent or, as physicists say, that are covariant. According to this author one then has to turn to theories taking the landscape of ridges and ruts, and the total grey-values in between as a finite CW-complex. The finite CW-complex structure can savely be subjected to morphisms defined in terms of the valencies and the couplings on the CW-complex. One might conjecture that depending on the particular chosen paradigm to achieve a task certain types of dynamical processes will survive, whereas others definitively will fade out. First studies and experiments confirm that certain types of dynamic scale-space paradigms lead to recurrence of certain topological image formation characteristics, whereas others definitively yield irreversibly trivial ones. Further research in dynamic scale-space theories with respect to CW-complexes representing image formation that conserve certain topological aspects should be under way by now. The outcomes of such a research might have some spin-off in autonomous system research, cognitive sciences and the field of artificial intelligence.

References

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