TIME-SCALE ANALYSIS OF DETERMINISTIC SIGNALS CORRUPTED BY ZERO-MEAN MULTIPLICATIVE NOISE

Marie CHABERT, Jean-Yves TOURNERET and Francis CASTANIE
2, rue Charles Camichel BP 7122
31071 Toulouse Cedex 7
email: chabert@len7.enseeiht.fr

ABSTRACT

This work applies the Continuous Wavelet Transform (CWT) to Abrupt Change (AC) detection in zero-mean multiplicative noise. An AC signature can be defined because of CWT translation invariance property. The problem is then signature detection in the time-scale domain. A contrast criterion is defined as a second-order measure of performance. This contrast depends on the first and second order moments of the multiplicative process CWT. An optimal wavelet maximizing the contrast is derived for an ideal step in white multiplicative noise. In this fundamental case, the Signal to Noise Ratios (SNR) in the time domain and in the time-scale domain are compared. A scale exists above which SNR is larger on the CWT maxima curve than in the time domain. An asymptotically optimal wavelet is derived for smoothed AC.

1. INTRODUCTION AND PROBLEM FORMULATION

Multiplicative noise models have been considered in many applications. These applications include image processing in systems using coherent radiation (radar, laser) [2] and random communication models (fading channels) [9]. For object contour extraction or for segmentation purpose, these applications require the detection and estimation of abrupt change (AC) in the observed process moments [1]. This paper applies the Continuous Wavelet Transform (CWT) to AC detection in zero-mean multiplicative noise. The detection algorithms are different whether the multiplicative noise is zero-mean or not. The CWT has been shown to be an effective tool for the AC detection problem when the noise is non-zero mean [3]. The optimal Neyman-Pearson detector and two suboptimal detectors based on the time-scale signature have been successfully studied [4]. Unfortunately, these algorithms fail when the noise is zero-mean.

This paper proposes to use a non-linear preprocessing followed by a time-scale detector for the AC detection problem, when the multiplicative noise is zero-mean. The quadratic non-linearity has been chosen for simplicity. However, other non-linearities could be studied in a similar way. The detection problem expresses as a simple binary hypotheses testing:

Under the null hypothesis H_0, the observed process $y(t)$ is a white noise with variance σ_x^2:

$$y(t) = x(t)$$

Under the hypothesis H_1, the observed process $y(t)$ is modeled by:

$$\begin{aligned} y(t) &= x(t)s(t) = x(t)\left[1 + Af\left(\frac{t-t_0}{a_0}\right)\right] \\ &= x(t)\left[1 + Af_{\circ}(t)\right] \\ t &\in \Omega,\ t_0 \in \Omega, A \geq 0,\ a_0 > 0 \end{aligned}$$

where Ω is the observation interval. $s(t)$ is a deterministic signal with parameter vector $\underline{\theta} = (A, t_0, a_0)^t$ (amplitude, instant, dilation). $s(t)$ corresponds to a transition from 1 to $1+A$. f characterizes the transition shape. f is assumed positive, bounded and satisfies:

$$\begin{aligned} f(t) &= 0 \text{ for } t \leq -1 \\ f(t) &= 1 \text{ for } t \geq 1 \end{aligned}$$

Note that $s(t)$ tends to an ideal step with amplitude A at position t_0 when $a_0 \longrightarrow 0$.

Section 2 analyzes the AC in the time-scale domain when noise is multiplicative. First and second order moments are derived under both hypotheses. The CWT translation invariance property allows a multiplicative AC signature to be defined. Section 3 proposes the complementary deflection as a contrast measure in the time-scale domain. It measures wavelet ability to separate the two hypotheses. The complementary deflection is equivalent to a Signal to Noise

Ratio (SNR) in the time-scale domain, under hypothesis H_1. Comparison of the SNR in the time and time-scale domains displays the CWT capabilities for detection in the multiplicative context. The SNR in the time-scale domain increases with the scale on the CWT maxima curve, whereas it is constant in the time domain. This section derives an optimal wavelet maximizing the SNR, for an ideal step on the CWT maxima curve. The minimum scale for which the SNR is larger on this curve than in the time domain is then determined. This wavelet is shown to be asymptotically optimal for any smoothed AC. Simulations and conclusions are reported in section 4 and 5.

2. TIME-SCALE ANALYSIS

The CWT of $y(t)$ is defined by:

$$\begin{aligned} C_y(a,\tau) &= \int_{-\infty}^{+\infty} y(t)\psi^*_{a,\tau}(t)\,dt \qquad (1) \\ \text{with} \quad \psi_{a,\tau}(t) &= a^{-1/2}\psi\left(\frac{t-\tau}{a}\right) \\ a &\in \mathbb{R}^*,\ \tau \in \mathbb{R} \end{aligned}$$

The analyzing function family $\{\psi_{a,\tau}\}_{a\in\mathbb{R}^*,\ \tau\in\mathbb{R}}$ is constructed by dilation and translation of a function ψ called the mother wavelet (a is the dilation parameter, τ is the translation parameter). If ψ satisfies the admissibility condition (which can be expressed as $\int_{-\infty}^{+\infty}\psi(t)dt = 0$ when the Fourier Transform of ψ is continuous) the transform admits a reconstruction formula [5]. This study is restricted to real normalized wavelets with symmetrical bounded support $\left[\frac{-\Delta t}{2}, \frac{\Delta t}{2}\right]$:

$$\int_{-\frac{\Delta t}{2}}^{+\frac{\Delta t}{2}} \psi(t)^2 dt = 1 \qquad (2)$$

The CWT is invariant with respect to translation and dilation of the original signal. This property is used in many detection and classification applications [8].

The CWT displayed good properties for the detection of AC, in **non-zero mean** multiplicative noise. The CWT emphasized the change of the observed process mean value. This change generated a signature in the time-scale domain. This signature was defined as the CWT mean value [6] and expressed as [3]:

$$\begin{aligned} E\left[C_y(a,\tau) \mid H_1\right] &= m_x A \int_{t_0-a_0}^{\tau+a\frac{\Delta t}{2}} f_0(t)\psi^*_{a,\tau}(t)\,dt \\ \text{where } m_x &= E\left[x(t)\right] \qquad (3) \end{aligned}$$

Eq. (3) shows that the AC detection problem is equivalent to a signature detection problem in the time-scale domain. Unfortunately, the signature based detection fails when noise is zero-mean, since the signature equals zero. This paper proposes to use a quadratic preprocessing followed by a time-scale detector.

Denote $m_i(a,\tau)$ and $\sigma_i^2(a,\tau)$ the mean and variance of $C_{y^2}(a,\tau)$ under hypothesis H_i, $i = 0, 1$. The admissibility condition implies that:

$$\begin{aligned} m_0(a,\tau) &= m_{x^2} \int_{\tau-a\frac{\Delta t}{2}}^{\tau+a\frac{\Delta t}{2}} \psi^*_{a,\tau}(t)\,dt = 0 \\ m_1(a,\tau) &= A m_{x^2} \int_{t_0-a_0}^{\tau+a\frac{\Delta t}{2}} \left[2f_0(t) + Af_0^2(t)\right]\psi_{a,\tau}(t)\,dt \end{aligned} \qquad (4)$$

The signature is conic and points to the AC occurrence time t_0. Fig's 1 and 3 show the step and ramp signatures for the symmetrical Haar wavelet. The symmetrical Haar wavelet is defined by:

$$\psi(t) = \begin{cases} \frac{1}{\sqrt{\Delta t}} & \text{if } -\frac{\Delta t}{2} \le t < 0 \\ -\frac{1}{\sqrt{\Delta t}} & \text{if } 0 \le t < \frac{\Delta t}{2} \\ 0 & \text{otherwise} \end{cases} \qquad (5)$$

This signature is embedded in noise in the time-scale plane. The variance of $C_{y^2}(a,\tau)$ satisfies the following equations:

$$\begin{aligned} \sigma_0^2(a,\tau) &= \sigma_{x^2}^2 \\ \sigma_1^2(a,\tau) &= \sigma_{x^2}^2 \int_{-\infty}^{+\infty} s(t)^4\, \psi^2_{a,\tau}(t)\,dt \end{aligned} \qquad (6)$$

Eq. 4 show that the AC detection problem can be studied using the pseudo-observation $C_{y^2}(a,\tau)$. The CWT associated with the quadratic non-linearity is a transition between the observation and the decision. The transition space is the space of the process squared CWT's. A contrast criterion allows to order the set of applications from the observation to the pseudo-observations [7].

3. CONTRAST IN THE TIME-SCALE DOMAIN

3.1. DEFINITION

A contrast criterion well-matched to the detection problem has to be determined. Define a contrast $\gamma_f(a,\tau)$ associated to the statistic C_{y^2}, at each point (a,τ) of the time-scale plane as follows [7]:

$$\gamma_f(a,\tau) = \frac{\left|m_1(a,\tau) - m_0(a,\tau)\right|^2}{Var_\alpha\left[C_{y^2}(a,\tau)\right]} \qquad (7)$$

In (7), $Var_\alpha\left[C_{y^2}(a,\tau)\right]$ is the variance corresponding to the mixing distribution:

$$p_\alpha(x) = (1-\alpha)\,p_0(x) + \alpha p_1(x) \qquad \text{with } \alpha \in [0,1] \qquad (8)$$

where $p_0(.)$ and $p_1(.)$ are the distributions of $C_{y^2}(a,\tau)$ under hypotheses H_0 and H_1. The criterions obtained for $\alpha = 0$ and $\alpha = 1$ are usually called deflection and complementary deflection. The choice of parameter α depends on the first and second order moments of $C_{y^2}(a,\tau)$ under hypotheses H_0 and H_1. Eq. (6) leads to:

$$\sigma_0^2(a,\tau) < \sigma_1^2(a,\tau) \qquad \forall \tau \in \Omega$$

Consequently, the complementary deflection criterion is a more restrictive contrast measure than the currently used deflection criterion. The complementary deflection criterion expresses as:

$$\gamma_f(a,\tau) = \frac{m_1(a,\tau)^2}{\sigma_1^2(a,\tau)}$$

Denoting $C = \frac{m_{x^2}^2}{\sigma_{x^2}^2} A^2$, the complementary deflection criterion becomes:

$$\gamma_f(a,\tau) = C \frac{\left[\int_{t_0-a_0}^{\tau+a\frac{\Delta t}{2}} \left[2f_0(t) + Af_0^2(t)\right] \psi_{a,\tau}(t)\, dt\right]^2}{\int_{-\infty}^{+\infty} s(t)^4 \psi_{a,\tau}^2(t)\, dt} \tag{9}$$

This contrast expresses as a SNR in the time-scale domain, under hypothesis H_1. It can be maximized with respect to the wavelet. Next section derives an optimal wavelet maximizing $\gamma_f(a,\tau)$ for an ideal step. The SNR on the CWT maxima curve is then shown to be larger than the SNR in the time domain, for large scales.

3.2. OPTIMAL WAVELET FOR AN IDEAL STEP

The optimal wavelet is obtained by maximization of the contrast criterion under three constraints.

The first constraint is the admissibility condition. The second is the wavelet normalization (2). For joint detection-estimation purpose, the conic AC signature modulus is expected to be maximum for $\tau = t_0$, at each scale. This provides an unbiased AC occurrence time estimate and an easy interpretation of the representation. This property leads to the third constraint:

$$m_1(a,t_0) > m_1(a,\tau) \qquad \forall \tau \neq t_0 \quad \forall a \in \mathbb{R}^*$$

which can be expressed as (for an ideal step):

$$h(0) > h(x) \qquad \forall x \neq 0$$

with $h(x) = \left(\int_x^{\frac{\Delta t}{2}} \psi(t)dt\right)^2$. The first derivative of $h(x)$ is:

$$h'(x) = -2\psi(x) \int_x^{\frac{\Delta t}{2}} \psi(t)dt$$

A sufficient condition for the third constraint is that the sign of ψ must be constant on $\left[0, \frac{\Delta t}{2}\right]$ and opposite on $\left[-\frac{\Delta t}{2}, 0\right]$. Note that if ψ is continuous, $\psi(0) = 0$.

When $f(t)$ is an ideal step, $\gamma_f(a,\tau)$ reduces to:

$$\gamma_{step}(a,\tau) = \frac{aC(A+2)^2 h\left(\frac{t_0-\tau}{a}\right)}{1 + \left((1+A)^4 - 1\right) \int_{\frac{t_0-\tau}{a}}^{\frac{\Delta t}{2}} \psi^2(t)\, dt} \tag{10}$$

A large SNR is required at the AC occurrence time t_0 in order to estimate this parameter. For $\tau = t_0$ and the symmetrical Haar wavelet, (10) leads to:

$$\gamma_{step}(a,t_0) = \frac{aC(A+2)^2 h(0)}{1 + \left((1+A)^4 - 1\right) \int_0^{\frac{\Delta t}{2}} \psi^2(t)\, dt} \tag{11}$$

The Cauchy-Schwarz inequality leads to:

$$h(0) \leq \int_0^{\frac{\Delta t}{2}} dt \int_0^{\frac{\Delta t}{2}} \psi^2(t)\, dt$$

Hence:

$$\gamma_{step}(a,t_0) \leq \frac{aC(A+2)^2 \frac{\Delta t}{2}}{\left[\int_0^{\frac{\Delta t}{2}} \psi^2(t)\, dt\right]^{-1} + (1+A)^4 - 1} \tag{12}$$

The equality in (12) is obtained when $\psi(x)$ is constant over $\left[0, \frac{\Delta t}{2}\right]$. Consequently the wavelet defined by:

$$\begin{cases} \psi(x) = C_\psi \in \mathbb{C} & \forall x \in \left[0, \frac{\Delta t}{2}\right] \\ \psi(x) = -C_\psi & \forall x \in \left[-\frac{\Delta t}{2}, 0\right] \end{cases}$$

belongs to the class of solutions, C_ψ is chosen such that ψ is normalized and hence verifies (2) : $C_\psi = \frac{1}{\sqrt{\Delta t}}$. This wavelet is finally the Haar wavelet with symmetrical support.

For $\tau = t_0$ and the symmetrical Haar wavelet, (11) leads to:

$$\gamma_{step}(a,t_0) = \frac{aC(A+2)^2 \Delta t}{2\left(1 + (1+A)^4\right)} \tag{13}$$

The SNR is proportional to the scale a for $\tau = t_0$. This property is still valid for $\tau \neq t_0$ and large scales (when $\frac{t_0-\tau}{a}$ goes to 0). This highlights the interest of working in the time-scale domain. In this simple case, the SNR in the time domain and in the time-scale domain can be compared.

In order to appreciate CWT contribution, the process $y^2(t)$ is studied in the time domain. The SNR is constant in the time domain, under both hypotheses:

$$\begin{aligned} \left[SNR(t) \mid H_i\right] &= \frac{E\left[y^2(t)\right]^2}{var\left[y^2(t)\right]} & (14) \\ &= \frac{m_{x^2}}{\sigma_{x^2}^2} \qquad \forall t \in \Omega & (15) \end{aligned}$$

The SNR in the time domain depends on the noise statistics and is independent of the signal $s(t)$. The SNR is greater in the time-scale domain than in the time domain, if:

$$a \geq a_{\min} = \frac{2\left(1+(1+A)^4\right)}{\Delta t\,(A^2+2A)^2} \tag{16}$$

This implies a minimum length for the observation interval: $L_{\min} = a_{\min}$.

3.3. ASYMPTOTIC OPTIMAL WAVELET FOR A SMOOTHED AC

A smoothed AC is now considered. The SNR $\gamma_f(a,\tau)$ can be maximized for $\tau = t_0$ in order to estimate the transition location:

$$\gamma_f(a,t_0) = \frac{aC\,(A+2)^2\,(I_1(a)+J_1(a))^2}{1+\left((1+A)^4-1\right)[I_2(a)+J_2(a)]} \tag{17}$$

with:

$$I_k(a) = \int_{-\frac{a_0}{a}}^{\frac{a_0}{a}} f^{2k}\left(\frac{at}{a_0}\right)\psi^k(t)\,dt \qquad k=1,2$$

$$J_k(a) = \int_{\frac{a_0}{a}}^{\frac{\Delta t}{2}} \psi^k(t)\,dt \qquad k=1,2$$

Since f and ψ are assumed bounded:

$$\lim_{a\longrightarrow+\infty} I_k(a) = 0 \qquad k=1,2$$

$$\lim_{a\longrightarrow+\infty} J_k(a) = \int_0^{\frac{\Delta t}{2}} \psi^k(t)\,dt \qquad k=1,2$$

Finally: $\gamma_f(a,t_0) \underset{a\longrightarrow+\infty}{\sim} \gamma_{step}(a,t_0)$. Consequently, the symmetrical Haar wavelet is asymptotically optimal for smoothed AC detection problem. It follows that the CWT is asymptotically robust to the AC shape.

4. SIMULATIONS

Numerous simulations have been performed to validate the previous results. For the following experiment, the transition $f(t)$ is a step (at $t_0 = 500$ with amplitude $A = 0.4$) or a ramp ($f(t) = t$, $t \in [-1,1]$, $A = 0.4$, $a_0 = 100$, $t_0 = 500$). The noise $x(t)$ is white Gaussian with variance $\sigma_x^2 = 1$. Fig's. 1 and 3 show the ideal signatures in both cases. The multiplicative AC signature cross-section (corresponding to fixed scales) is proportional to the wavelet integral, in the case of an ideal step. The symmetrical Haar wavelet leads to triangular cross sections. Fig's 2 and 4 represent the noisy CWT of $y^2(t)$. The CWT is computed for scale varying from 200 to 350 with the symmetrical Haar wavelet. The conic signature is pointing to the AC occurrence time $t_0 = 500$, in both cases. Note that the cone is smoother and of lower amplitude, for a ramp than for a step. However, it emerges from noise for large scales, in both cases.

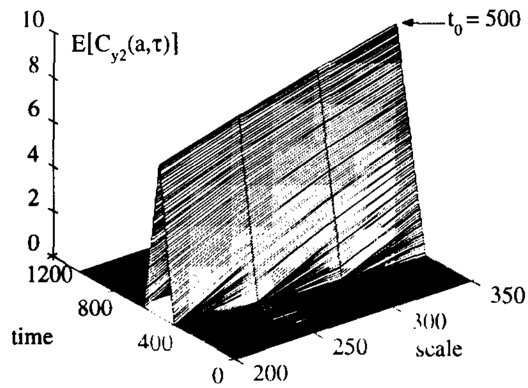

Fig. 1 : Signature of a step ($A = 0.4, t_0 = 500$) with symmetrical Haar wavelet

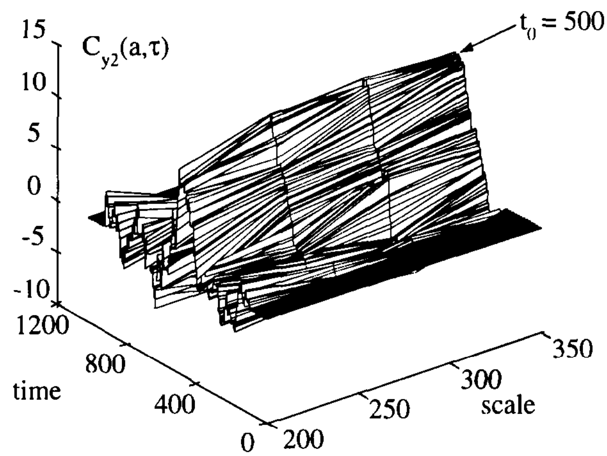

Fig. 2 : CWT of a step ($A = 0.4, t_0 = 500$) embedded in multiplicative noise ($\sigma_x = 1$) with symmetrical Haar wavelet

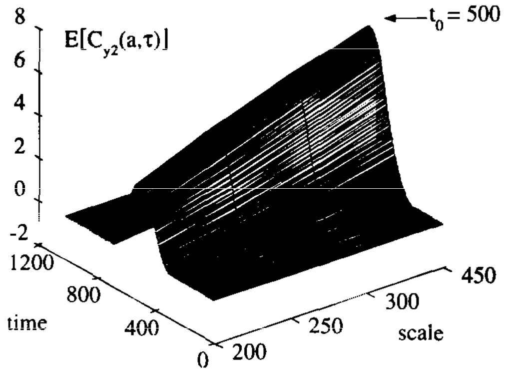

Fig. 3 : Signature of a ramp ($A = 0.4, t_0 = 500, a_0 = 200$) with symmetrical Haar wavelet

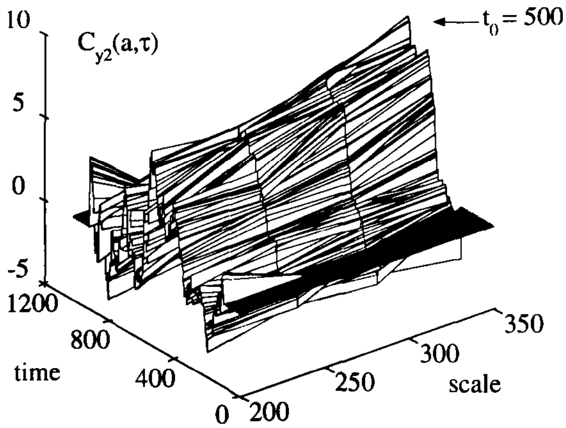

Fig. 4 : *CWT* of a ramp ($A = 0.4, t_0 = 500, a_0 = 200$) embedded in multiplicative noise ($\sigma_x = 1$) with symmetrical Haar wavelet

5. CONCLUSION

Abrupt changes corrupted by zero-mean multiplicative noise were studied using the continuous wavelet transform and a quadratic non-linearity. The first and second order moments of the observed process continuous wavelet transform were derived. These moments allowed to define an abrupt change signature in the time-scale domain. The complementary deflection was chosen as a contrast criterion in the time-scale domain. The Haar wavelet was shown to maximize the contrast on the continuous wavelet transform maxima curve, for an ideal step and a white noise. The signal to noise ratio on the continuous wavelet transform maxima curve (for some minimum scale) was larger than in the time domain. For the smoothed abrupt change, the Haar wavelet was asymptotically optimal on the continuous wavelet transform maxima curve. The continuous wavelet transform is an effective tool for abrupt change detection, since it is asymptotically robust to the transition shape.

6. REFERENCES

[1] M. Basseville and I. V. Nikiforov, *Detection of Abrupt Changes: Theory and Application.* Englewood Cliffs, NJ: Prentice-Hall, 1993.

[2] A. C. Bovik, "On detecting Edges in Speckle Imagery," IEEE Trans. A.S.S.P., vol. 36, n°10, pp. 1618-1627, Oct. 1988.

[3] M. Chabert, J-Y. Tourneret and F. Castanié, "Additive and Multiplicative Jump Detection using Continuous Wavelet Transform," Proc. ICASSP'96, Atlanta, 1996.

[4] M. Chabert, J-Y. Tourneret and F. Castanié, "Performance of an Optimal Multiplicative Jump Detector based on the Continuous Wavelet Transform," Proc. EUSIPCO'96, Trieste, 1996.

[5] I. Daubechies, *Ten Lectures on Wavelets,* CBMS-NSF Regional Conference Series In applied Mathematics, 1992.

[6] A. Denjean and F. Castanié, "Mean Value Jump Detection: a Survey of Conventional and Wavelet Based Methods," dans Wavelet Theory, Algorithms and Applications,C. K. Chui, L. Montefuso and L. Puccio (eds.), Academic Press, Inc., 1994.

[7] B. Picinbono and P. Duvaut, "Detection and Contrast," Stochastic Processes in Underwater Acoustics, Lecture Notes in Control and Information Sciences (vol 85), Baker CR et Al (Eds), 1986.

[8] E. P. Simoncelli, W. T. Freeman, E. H. Adelson and D. J. Heeger, "Shiftable Multiscale Transforms," *IEEE Trans Information Theory,* vol.38, n°2, pp.587-607, 1992.

[9] H. L. Van Trees, Detection, Estimation, and Modulation Theory, John Wiley & Sons, 1968.