PARALLELS BETWEEN GRANULOMETRIC AND FOURIER TRANSFORMS

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ABSTRACT

Granulometric transforms are used for image (texture) classification and characterization of optimal morphological (granulometric) bandpass filters. The present paper discusses parallels between these two applications of granulometric transforms for random sets and two corresponding applications of Fourier transforms, classification of random arcs by Fourier descriptors and characterization of the Wiener filter via power spectral densities.

1. INTRODUCTION

Granulometries and their size distributions were introduced by Matheron to characterize sieving filters and their effects on random sets [1]. The present paper discusses more recent developments concerning how granulometries induce transforms on random sets and how these transforms are used for two standard image processing tasks: classification via feature vectors and design of optimal filters for the signal-union-noise model. It also discusses parallels between granulometric and Fourier feature-based classification, and between the roles played by granulometric and Fourier transforms in designing optimal filters for the signal-union-noise model and the signal-plus-noise model, respectively. We begin by outlining the basic definitions and properties of multivariate granulometries.

Let $\mathbf{G} = \{B_1, B_2, \ldots, B_n\}$ be a collection of compact, convex sets such that no set in $\mathbf{G}$ is open relative to another set in $\mathbf{G}$. For $\mathbf{t} = (t_1, t_2, \ldots, t_n)$, $t_i > 0$ for $i = 1, \ldots, n$, define the set operator $\Psi_{\mathbf{t}}$ by

$$\Psi_{\mathbf{t}}(S) = \bigcup_{i=1}^{n} S \circ t_i B_i \tag{1}$$

As a union of openings, $\Psi_{\mathbf{t}}$ is a (multiparameter) τ-opening with base $\{t_1 B_1, \ldots, t_n B_n\}$. For any $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ for which there exists $t_i = 0$, define $\Psi_{\mathbf{t}}(S) = S$. $\{\Psi_{\mathbf{t}}\}$ is called an n-dimensional *multivariate granulometry* with generator $\mathbf{G}$. The *multivariate size distribution* for S is defined by $\Omega(\mathbf{t}) = \nu[\Psi_{\mathbf{t}}(S)]$, ν denoting volume, the *inverted size distribution* by $H(\mathbf{t}) = \nu[S] - \Omega(\mathbf{t})$, and the *multivariate pattern spectrum* by $\Phi(\mathbf{t}) = 1 - \Omega(\mathbf{t})/\nu[S]$. $\Omega(\mathbf{t})$ is a probability distribution function [2] and, for any nonnegative-integer vector $\mathbf{k} = (k_1, k_2, \ldots, k_n)$, the $\mathbf{k}$th moment of Φ is given by the Stieltjes integral

$$\mu^{(\mathbf{k})}(S) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} t_1^{k_1} \cdots t_n^{k_n} d\Phi(t_1, \ldots, t_n) \tag{2}$$

The class of finite-generator Euclidean (univariate) granulometries whose generator sets are compact and convex is a subclass of the multivariate granulometries because, for $\mathbf{t} = (t, t, \ldots, t)$, $\{\Psi_{\mathbf{t}}\}$ is the Euclidean granulometry $\{\Psi_t\}$.

2. CLASSIFICATION

Treating S as a random set, the mapping $S \rightarrow \Phi_S$ provides a transform yielding a multivariate random function. For classification, we use the random-vector transform $S \rightarrow \{\mu^{(\mathbf{k})}(S)\}$, where some finite set of moments is used to produce a feature vector. Granulometric feature vectors have been used for texture classification (including gray-scale granulometric features) [2]-[4]. Most applications have involved univariate granulometric moments; however, mulivariate-granulometric-moment features are more discriminatory.

To see the parallels between granulometric-moment features and standard Fourier descriptors, consider a random planar arc having the parametric description $\Gamma(t) = (X(t), Y(t))$. Restricting ourselves to cosine transforms, $X(t)$ has Fourier coefficients given by

$$A_n = \frac{2}{T} \int_0^T X(t) \cos \frac{2n\pi t}{T} dt \tag{3}$$

(T being arc length) and the cosine coefficients C_n for $Y(t)$ are defined similarly. For fixed m, the random vectors $(A_0, A_1, \ldots, A_m)$ and $(C_0, C_1, \ldots, C_m)$ taken together form a feature vector for Γ. For Gaussian maximum-likelihood classification, the covariance matrix of the combined vector plays a central role. For

the cosine transform, a second moment is of the form

$$E\left[A_n C_k\right] = \frac{4}{T^2}\int_0^{\pi}\int_0^{\pi} R_{XY}(t,s)\cos\frac{2n\pi t}{T}\cos\frac{2k\pi s}{T}dtds \tag{4}$$

where $R_{XY}(t,s)$ is the autocorrelation function for the coordinate functions.

For a univariate granulometric classifier using moments $\mu^{1,1}, \mu^{1,2}, \ldots, \mu^{1,m}$ from granulometry $\{\Psi_{1,t}\}$ and moments $\mu^{2,1}, \mu^{2,2}, \ldots, \mu^{2,m}$ from granulometry $\{\Psi_{2,t}\}$, a second moment looks like

$$E\left[\mu^{1,n}\mu^{2,k}\right] = \int_0^{\infty}\int_0^{\infty} R_{\Phi_1'\Phi_2'}(t,s)t^n s^k\, dtds \tag{5}$$

In both cases, the second-order moments of the features are given in terms of the corresponding two-dimensional transform of the autocorrelation function of the representative functions. There are differing interpretations depending on the degree to which the coordinate functions and cosine (Fourier) transform represent the random arc and the degree to which the pattern spectra and their moments represent the random set. An advantage of granulometric representation is that individual granulometric features are often asymptotically normal with known asymptotic distributions [5, 6]. An advantage of Fourier descriptors is that, except for compression resulting from choosing a finite number of coefficients, the Fourier transform can be considered to be invertible for practical modeling, whereas the granulometric transform is not generally invertible and general conditions regarding its invertibility are unknown.

3. RECONSTRUCTIVE MULTIPARAMETER OPENINGS

The signal-union-noise model consists of a signal random set S, noise random set N, observed set $S \cup N$, and filter Ψ. $\Psi(S \cup N)$ estimates S and the goodness of the estimate is measured by some probabilistic error criterion. We consider the granular model $S \cup N$, where

$$S = \bigcup_{i=1}^{I} C[\mathbf{s}_i] + x_i, \quad N = \bigcup_{j=1}^{J} D[\mathbf{n}_j] + y_j \tag{6}$$

I and J are random natural numbers; $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_I$ and $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_J$ are identically distributed to random vectors $\mathbf{s} = (s_1, s_2, \ldots, s_m)$ and $\mathbf{n} = (n_1, n_2, \ldots, n_m)$, respectively; $C[\mathbf{s}_i]$ and $D[\mathbf{n}_j]$ are random compact grains (connected components) governed by $\mathbf{s}_i$ and $\mathbf{n}_j$, respectively, and identically distributed with the primary grains $C[\mathbf{s}]$ and $D[\mathbf{n}]$; and x_i and y_j are random points governing grain locations (translations) under the constraint of grain disjointness.

If signal components are probabilistically larger than noise components and we focus on the decision to pass or not pass a component in the observed image, then it is appropriate to consider optimization of reconstructive granulometric filters. If $\{\Psi_{\mathbf{t}}\}$ is a multiparameter granulometry, then the corresponding reconstructive granulometry $\{\Lambda_{\mathbf{t}}\}$ is defined componentwise by fully passing any component not entirely eliminated by $\Psi_{\mathbf{t}}$ and eliminating any component eliminated by $\Psi_{\mathbf{t}}$. Optimization with respect to a reconstructive granulometry is achieved by finding $\mathbf{t}$ to minimize the expected error $E[\nu[\Lambda_{\mathbf{t}}(S \cup N) \triangle S]]$, $\triangle$ denoting symmetric difference. The optimal filter is a reconstructive multiparameter τ-opening.

The signal and noise pass sets are defined by $\mathbf{M}_S = \mathbf{M}_{C[\mathbf{s}]} = \{\mathbf{t}\colon \Lambda_{\mathbf{t}}(C[\mathbf{s}]) = C[\mathbf{s}]\}$ and $\mathbf{M}_N = \mathbf{M}_{D[\mathbf{n}]} = \{\mathbf{t}\colon \Lambda_{\mathbf{t}}(D[\mathbf{n}]) = D[\mathbf{n}]\}$, respectively. Filter error for the parameter $\mathbf{t}$ is given by

$$\begin{aligned} e(\mathbf{t}) &= E[I]\int\cdots\int_{\{\mathbf{s}:\mathbf{t}\notin \mathbf{M}_{C[\mathbf{s}]}\}} \nu[C[\mathbf{s}]] f_S(\mathbf{s})d\mathbf{s} \\ &+ E[J]\int\cdots\int_{\{\mathbf{n}:\mathbf{t}\in \mathbf{M}_{D[\mathbf{n}]}\}} \nu[D[\mathbf{n}]] f_N(\mathbf{n})d\mathbf{n} \end{aligned} \tag{7}$$

where $E[I]$ and $E[J]$ are the expected numbers of signal and noise grains, respectively, and f_S and f_N are the multivariate densities for the random vectors $\mathbf{s}$ and $\mathbf{n}$, respectively [7].

The multivariate size distribution for the signal S and its mean are given by

$$\Phi_S(\mathbf{t}) = \sum_{\{i:\mathbf{t}\in \mathbf{M}_{C[\mathbf{s}_i]}\}} \nu[C[\mathbf{s}_i]] \tag{8}$$

$$E\left[\Phi_S(\mathbf{t})\right] = E[I]\int\cdots\int_{\{\mathbf{s}:\mathbf{t}\in \mathbf{M}_{C[\mathbf{s}]}\}} \nu[C[\mathbf{s}]] f_S(\mathbf{s})d\mathbf{s} \tag{9}$$

and similarly for the noise N. Hence, with $\mathbf{H}_S$ and $\mathbf{H}_N$ denoting the expectations of the inverted signal and noise size distributions, and A denoting expected image area, filter error is given by

$$e[\mathbf{t}] = A - E[\Phi_S(\mathbf{t})] + E[\Phi_N(\mathbf{t})] = \mathbf{H}_S(\mathbf{t}) + A - \mathbf{H}_N(\mathbf{t}) \tag{10}$$

Assuming that $\mathbf{H}_S$ and $\mathbf{H}_N$ are sufficiently regular (which is often the case),

$$\begin{aligned} e(\mathbf{t}) = &\int_0^{t_1}\cdots\int_0^{t_n} \frac{\partial^n \mathbf{H}_S(u_1,\ldots,u_n)}{\partial u_1\ldots\partial u_n}du_1\ldots du_n \\ + &\; A - \int_{t_1}^{\infty}\cdots\int_{t_n}^{\infty} \frac{\partial^n \mathbf{H}_N(u_1,\ldots,u_n)}{\partial u_1\ldots\partial u_n}du_1\ldots du_n \end{aligned} \tag{11}$$

For a univariate granulometry, the random sets $\mathbf{M}_S$ and $\mathbf{M}_N$ are replaced by the *granulometric sizes* $M_S = sup\{r : \Lambda_r(C[\mathbf{s}]) = C[\mathbf{s}]\}$ and $M_N = sup\{r : \Lambda_r(D[\mathbf{n}])$

$= D[\mathbf{n}]\}$, the domains of integration in Eq. 7 become the regions $r > M_S$ and $r \leq M_N$, and

$$e[t] = \int_0^t \mathbf{H}'_S(u)du + \int_t^\infty \mathbf{H}'_N(u)du \qquad (12)$$

if $\mathbf{H}_S$ and $\mathbf{H}_N$ are sufficiently regular. Suppose we wish to choose t to minimize the error. If it happens that there exists t_0 such that $\mathbf{H}'_S(t) \leq \mathbf{H}'_N(t)$ for all $t \leq t_0$ and $\mathbf{H}'_S(t) \geq \mathbf{H}'_N(t)$ for all $t \geq t_0$, $e[t]$ is minimized for $t = t_0$. If there does not exist such a point t_0, one may still be able to find a minimal error over all reconstructive univariate τ-openings; however, it is better to use a reconstructive granulometric bandpass filter.

4. GRANULOMETRIC BANDPASS FILTERS

Consider an upper semicontinuous, distributive Euclidean (univariate) granulometry $\{\Psi_t\}$, noting that, if $\{\Psi_t\}$ has a finite generator of compact, convex sets, then $\{\Psi_t\}$ is u.s.c. The *continuous granulometric spectrum* of set S relative to $\{\Psi_t\}$ is defined for $t \geq 0$ by

$$S_t = \bigcap_{h>0} \Psi_t(S) - \Psi_{t+h}(S) = \Psi_t(S) - \bigcup_{h>0} \Psi_{t+h}(S) \qquad (13)$$

Each S_t is called *a spectral component* of S. The spectral components partition S. For any t-interval I, the *spectral band* $G_S(I)$ of the image S determined by $\{\Psi_t\}$ is the union of the spectral components over I. A *countable-interval subset* of $[0,\infty)$ is a subset $\Pi \subset [0,\infty)$ that can be expressed as a countable union of disjoint intervals Π_i, where singleton point sets are intervals of length zero. Without loss of generality we assume that $i < j$ implies that, if $t \in \Pi_i$ and $r \in \Pi_j$, then $t < r$ and there exists $s \notin \Pi$ such that $t < s < r$. This assumption means that Π_i is to the left of Π_j and that Π_i and Π_j are separated by Π^c, which we denote by X. If $\{\Psi_t\}$ is an u.s.c. distributive Euclidean granulometry and Π is a countable-union subset, then the *continuous granulometric bandpass filter* Λ_Π is defined by

$$\Lambda_\Pi(S) = \bigcup_{t\in\Pi} S_t = \bigcup_{i=1}^{\infty} G_S(\Pi_i) \qquad (14)$$

where, the second union may be finite [8]. Π and X are the *pass* and *fail* sets for Λ_Π. X has a decomposition similar to that for Π: $X = \cup_j X_j$.

In general, $\mathbf{H}_S(t)$ is an increasing function of t and $\mathbf{H}_S(0) = 0$. We assume $\mathbf{H}_S$ is of bounded variation and continuous from the left. Then $\mathbf{H}_S$ is continuous except on at most a countable set, $\mathbf{H}_S$ is differentiable a.e., $\mathbf{H}'_S$ is integrable, and $\mathbf{H}_S$ possesses the Lebesgue decomposition $\mathbf{H}_S(t) = K_S(t) + A_N(t)$, where K_S is increasing, $K'_S(t) = 0$ a.e., A_S is increasing, A_S is absolutely continuous, A_S is differentiable a.e., and $A_S(t)$ is obtained by integrating $\mathbf{H}'_S$ from 0 to t. We assume the singular part K_S is a step function with a countable number of steps and denote the jump at t by $J_S(t)$.

The following error representation applies when the observed image is $S \cup N$, signal and noise are compact, and $S \cap N = \emptyset$: if countable-union pass and fail sets are decomposed into sets of intervals $\langle a_1, b_1\rangle$, $\langle a_2, b_2\rangle$, ... and $\langle c_1, d_1\rangle$, $\langle c_2, d_2\rangle$, ..., respectively (where angle brackets indicate that it does not matter whether or not the endpoints are included in the interval), then

$$\begin{aligned} e[\Lambda_\Pi] &= \sum_{i=1}^{\infty} \int_{a_i}^{b_i} \mathbf{H}'_N(u)du + \sum_{j=1}^{\infty} \int_{c_j}^{d_j} \mathbf{H}'_S(u)du \\ &+ \sum_{t\in\Pi} J_N(t) + \sum_{t\in X} J_S(t) \end{aligned} \qquad (15)$$

Let $D[S]$ and $D[N]$ denote the sets on which $\mathbf{H}_S$ and $\mathbf{H}_N$ are differentiable, respectively. The point sets at which A_S and A_N are differentiable include $D[S]$ and $D[N]$, respectively. K_S and K_N each have a countable number of jumps, their jump sets to be denoted by $J[S]$ and $J[N]$, respectively. For the u.s.c. distributive Euclidean granulometry $\{\Psi_t\}$ and the disjoint, compact signal-union noise model $S \cup N$, we define the $\{\Psi_t\}$-*induced pass set* $\Pi\langle\Psi\rangle$ by $t \in \Pi\langle\Psi\rangle$ if and only if one of the following three conditions is satisfied: (i) $t \in D[S] \cap D[N]$ and $A'_S(t) \geq A'_N(t)$; (ii) $t \in J[S] - J[N]$; (iii) $t \in J[S] \cap J[N]$ and $J_S(t) \geq J_N(t)$. The corresponding fail set is $X\langle\Psi\rangle = \Pi\langle\Psi\rangle^c$. The corresponding filter, Λ_Ψ, is called the $\{\Psi_t\}$-*induced filter*. Λ_Ψ is optimal in the sense that it has minimum error among all granulometric bandpass filters induced by countable-union subsets: if $\Pi\langle\Psi\rangle$ is a countable-union subset, then $e[\Lambda_\Psi] \leq e[\Lambda_\Pi]$ for any other countable-union subset Π [8]. For this theorem to make sense, $\Pi\langle\Psi\rangle$ must be a countable-union subset. Since $\mathbf{H}_S$ and A_S need only be differentiable a.e., we need a practical sufficient condition for $\Pi\langle\Psi\rangle$ to be a countable-union subset. If $\mathbf{H}_S$ and $\mathbf{H}_N$ possess continuous derivatives except on sets without limit points, then $\Pi\langle\Psi\rangle$ is a countable-union subset.

It is common for inverted-size-distribution means to have no singular part. In this case, the induced pass set is defined by

$$\Pi\langle\Psi\rangle = \{t : \mathbf{H}'_S(t) \geq \mathbf{H}'_N(t)\} \qquad (16)$$

If either derivative fails to exist or the derivatives are equal at t, then the choice of pass or fail set for t is irrelevant. Furthermore, if $\Pi\langle\Psi\rangle = \cup_i \langle a_i, b_i\rangle$, where $b_i < a_{i+1}$ for $i = 1, 2, \ldots$, then the induced optimal

filter is given by

$$\Lambda_{\Psi}(S) = \bigcup_{i=1}^{\infty} \Psi_{b_i} - \Psi_{a_i} \tag{17}$$

The bandpass nature of the filter is evident.

More can be said if we interpret the mean derivatives as generalized functions. Assume $\mathbf{H}_S$ and $\mathbf{H}_N$ are continuously differentiable except on sets without limit points. By definition of $\Pi\langle\Psi\rangle$, we can ignore points outdide $D_{S,N} = (D[S] \cap D[N]) \cup J[S] \cup J[N]$. At such points, $\mathbf{H}_S$ and $\mathbf{H}_N$ are continuous but do not possess derivatives. Under the hypothesis, all such points are isolated and their inclusion or lack of inclusion in $\Pi\langle\Psi\rangle$ has no effect on the error. Moreover, the jump points of K_S and K_N are isolated and therefore K_S and K_N can be represented as generalized functions. Hence, on $D_{S,N}$, $\mathbf{H}'_S$ has the representation

$$\mathbf{H}'_S(t) = \sum_{i=1}^{\infty} J_S[t_{S,i}]\delta(t - t_{S,i}) + A'_S(t) \tag{18}$$

where $J[S] = \{t_{S,1}, t_{S,2}, \ldots\}$. $\mathbf{H}'_N$ has a similar representation. With the usual interpretation of impulse functions and their intensities, $t \in \Pi\langle\Psi\rangle$ if and only if $\mathbf{H}'_S(t) \geq \mathbf{H}'_N(t)$, as in Eq. 16.

There is a clear analogy between granulometric bandpass filters and the Wiener filter in the wide-sense-stationary signal-plus-noise model with uncorrelated noise [8, 9], and this analogy is expressed via the characterization of Eq. 16. In the latter model the frequency response of the Wiener filter is given by $\mathbf{S}_S/(\mathbf{S}_S + \mathbf{S}_N)$, $\mathbf{S}_S$ and $\mathbf{S}_N$ being the power spectral densities of the signal and noise, respectively. A realization x is observed and the inverse Fourier transform

$$\mathcal{F}[\widehat{y}](\omega_1, \omega_2) = \frac{\mathbf{S}_S(\omega_1, \omega_2)}{\mathbf{S}_Y(\omega_1, \omega_2) + \mathbf{S}_N(\omega_1, \omega_2)} \mathcal{F}[x](\omega_1, \omega_2) \tag{19}$$

is taken as the Fourier transform of the signal estimate $\widehat{y}$. If the frequency response is rounded to 1 or 0, then the situation is akin to passing or not passing granulometric spectral components: a frequency component is passed if and only if $\mathbf{S}_S \geq \mathbf{S}_N$ and this binarization of the Wiener filter determines the passbands. Since the power spectral density is the Fourier transform of the covariance function, the optimal linear (Wiener) filter is determined by the covariance function, whereas the optimal granulometric bandpass filter is determined by the mean of the inverted size distribution.

5. CONCLUSION

We have summarized salient propositions from the theory of granulometric transforms and shown how application of these transforms corresponds to applications of Fourier transforms for both classification and optimal filter design. There are also similarities between adaptive granulometric filters [10] and adaptive linear filters (and neural networks); however, owing to space and our desire to focus on transform methodology, we have not discussed adaptive granulometric filters.

6. REFERENCES

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