NONLINEAR BACK PROJECTION FOR TOMOGRAPHIC IMAGE RECONSTRUCTION

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ABSTRACT

The objective of this paper is to investigate a new non-iterative paradigm for image reconstruction based on the use of nonlinear back projection filters. This method, which we call nonlinear back projection (NBP), attempts to directly model the optimal *inverse* operator through off-line training. Potential advantages of the NBP method include the ability to better account for effects of limited quantities and quality of measurements, image cross-section properties, and forward model non-linearities. We present some preliminary numerical results to illustrate the potential advantages of this approach and to illustrate directions for future investigation.

1. INTRODUCTION

In recent years, considerable effort has been put into the development Bayesian model-based approaches to tomographic image reconstruction[1, 2, 3]. While these methods can substantially improve reconstruction quality, these iterative methods can be computationally demanding, requiring at least 10 to 20 times the computation of filtered back projection. Even the best reconstruction algorithms may produce artifacts that can be visually identified in limited data problems. In fact, several techniques have been proposed to take advantage of restrictive *a priori* knowledge of the object to compensate for these limitations in nonlinear geometric reconstruction [4, 5, 6]. The fact remains that conventional filtered back projection (FBP) can produce surprisingly good quality reconstructions despite its limitation to simple averaging of projection data.

In this research, we propose a more direct and nonlinear approach to the Bayesian tomographic inverse problem. Rather than trying to develop an accurate forward model that can be inverted, our approach is to **directly model the inverse operator.** The goal is to develop an non-iterative Bayesian reconstruction method which requires computation comparable to conventional CBP methods, but achieves quality comparible to or better than that of current Bayesian methods.

The method we propose forms a back projected image cross-section by applying non-linear filters to the projected data. This method, which we call non-linear back projection, attempts to directly model the optimal *inverse* operator through off-line training of these non-linear filters using example training data. This direct approach to modeling of the inverse operator has a number of potential advantages which make it interesting:

Better modeling of image cross-section behavior - Current Bayesian models, such as MRF's are limited in their complexity by the difficulty of estimating model parameters. A direct model can be more effectively trained for the attributes of typical image cross-sections.

Better modeling of non-linear forward models - Non-linear forward models are difficult to incorporate in current Bayesian methods. This problem is avoided by direct modeling of the inverse operator.

Less computation - Since direct non-linear inversion is not iterative, it has potentially much lower computational requirements.

2. NON-LINEAR BACK PROJECTION METHOD

The nonlinear back projection tomography (NBP) method is illustrated in Fig. 1. Conventional CBP works, as shown in Fig. 1(a), by filtering the projections along a specific angle, θ, and then back projecting the result. Theoretically, this method yields perfect reconstruction with a continuum of data, but in practice it is well known to produce artifacts, and either overly smooth

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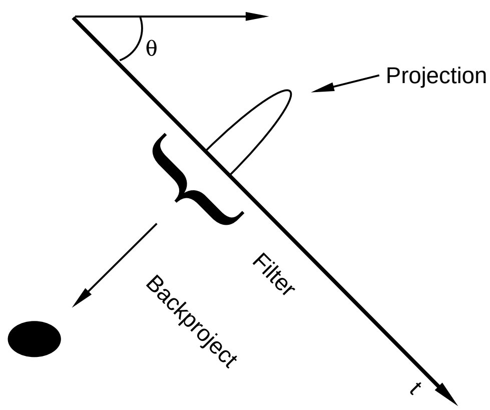
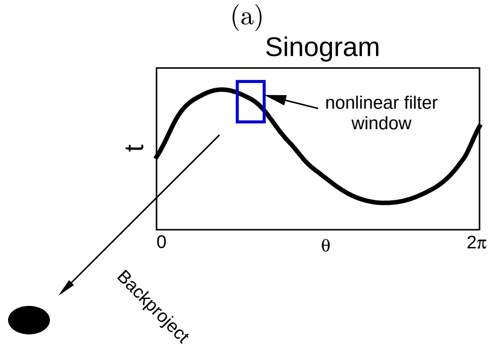


(a)



(b)

Figure 1: This figure illustrates that basic concept of nonlinear back projection. (a) Conventional CBP works by filtering the projections at each angle and then back projecting the result. (b) Nonlinear back projection applies a nonlinear filter to a window of data in the sinogram and then back projects this result.

or excessively noisy reconstructions. The NBP method works by applying a nonlinear filter to a window in the sinogram space as illustrated in Fig. 1(b). The behavior of this filter depends on the local characteristics of the sinogram. In addition, the filter's characteristics may depend on the specific pixel being back projected; however, we will not consider this dependence in the present investigation. The non-linear filter is then formed by combining M distinct linear filters, each of which is designed to minimize mean squared error for some class of input values. A related model has been proposed by Popat and Picard [7] for application in image restoration, and compression.

3. DERIVATION OF NBP ALGORITHM

Let Y be a vector containing all the sinogram data, and let X be a pixel in the unknown image cross-section. Furthermore, let Y_j be the vector of sinogram samples taken from the j^{th} window. In other words, j indexes the projection angle and Y_j contains the samples from the filter window of Fig. 1(b). Notice that the position of the window along both the t and θ coordinates will depend on the particular pixel being reconstructed. For simplicity, we will assume that the center pixel of the window falls precisely on the on the path integral required for back projection. In practice, the back projection is interpolated by a weighted combination of neighboring projection windows, but this does not substantively effect the resulting analysis.

The conventional CBP reconstruction may be expressed as

$$X = \sum_j \mathbf{F} Y_j$$

where $\mathbf{F}$ is the matrix which implements a filter along t, and the sum over j is the back projection operation. For conventional CBP, the matrix $\mathbf{F}$ does not depend on j, the projection angle. Moreover, most of the elements in $\mathbf{F}$ are zero since only the samples at a single angle are filtered.

For NBP, we will make the assumption that each set of samples, Y_j, has associated with it a discrete class, C_j, which takes on values between 0 and $M-1$. Instead of a single filter, we will have M filters denoted by $\mathbf{F}_c$ where $0 \leq c < M$. We will make two assumptions expressed in the following two equations.

$$\begin{aligned} E[X|Y,C] &= \sum_j \mathbf{F}_{C_j} Y_j \\ P\{C_j = c|Y\} &= f(c|Y_j)\ . \end{aligned}$$

The first equation states that given the class information, the minimum mean squared error (MMSE) estimate may be obtained by applying an appropriate filter to each window. The second equation states that the distribution of each class is only dependent on the pixels in the associated window. Using these two assumptions, we compute the MMSE estimator of X given Y.

$$
\begin{aligned}
E[X|Y] &= E[E[X|Y,C]|Y] \qquad (1)\\
&= E\left[\sum_j \mathbf{F}_{C_j} Y_j \middle| Y\right]\\
&= \sum_j E\left[\mathbf{F}_{C_j}|Y\right] Y_j\\
&= \sum_j \left(\sum_c \mathbf{F}_c f(c|Y_j)\right) Y_j
\end{aligned}
$$

Intuitively, this optimal estimator is formed by applying a spatially varying filter, $\sum_c \mathbf{F}_c f(c|Y_j)$ to the sinogram. Since the filter depends on the data in the window through $f(c|Y_j)$, this is actually a nonlinear filter. The image is then reconstructed by back projecting the nonlinearly filtered sinogram.

To apply this strategy, the filters $\mathbf{F}_c$ and the distribution $f(c|Y_j)$ must be estimated. To do this, we first rewrite (1) in the form

$$
E[X|Y] = \sum_c \mathbf{F}_c \left(\sum_j f(c|Y_j) Y_j\right) = \sum_c \mathbf{F}_c \bar{Y}_c
$$

where $\bar{Y}_c = \sum_j f(c|Y_j)Y_j$. By defining,

$$
\mathbf{F}_* = [\mathbf{F}_0, \mathbf{F}_1, \cdots, \mathbf{F}_{M-1}]
$$

$$
\bar{Y} = \begin{bmatrix} \mathbf{Y}_0 \\ \mathbf{Y}_1 \\ \vdots \\ \mathbf{Y}_{M-1} \end{bmatrix}
$$

the optimal filters may be easily computed as the least squares solution to

$$
\min_{\mathbf{F}_*} E\left[\left\|X - \mathbf{F}_* \bar{Y}\right\|^2\right]
$$

In order to compute the probabilities $f(c|Y_j)$, we use a Gaussian mixture model for Y_j so that

$$
p(y_j) = \sum_c p(y_j|c)\pi_c
$$

where $p(y_j|c)$ is a multivariate Gaussian distribution, and $\sum_c^{M-1} \pi_c = 1$. The parameters of $p(y_j|c)$ and the probabilities π_c may be estimated using the EM algorithm [8, 9, 10]. Given the mixture model, we have

$$
f(c|Y_j) = \frac{p(y_j|c)\pi_c}{\sum_c p(y_j|c)\pi_c} \; .
$$

4. EXPERIMENTAL RESULTS

In this section we present preliminary experimental results to illustrate the method of NBP. Figures 2 (a) and (b) show the synthetic phantom together with the filter back projection reconstruction. The cross section was reconstructed at a resolution of 128×128 from 16 uniformly spaced projection angles. Figure 2 (c) and (d) show the result of training and applying NBP with 18 and 60 clusters respectively. Notice that the NBP reconstructions have reduced artifacts because each of the filters is designed to smooth these errors. However, some sharpness is also lost along the edges of features. We believe that this sharpness can be recovered by allowing filters to vary depending on the spatial location of pixels in the image.

Figure 3 illustrates how the filters for various clusters vary. Each slice through the 3-D plot shows the filter values for a specific cluster. Notice that some filters are impulsive while others are smoothing functions.

Figure 4 shows how the mean squared error (MSE) varies with the number of clusters used in the NBP reconstruction. The MSE is normalized with respect to filtered back projection reconstruction; so an MSE of 1 is equal to that of filtered back projection. Notice that as the number of clusters is increased the MSE decreases because the projections with different behaviors can be treated differently. This effect is more noticeable when the MSE is computed on the entire image because of the large discontinuity generated at the support boundary of the object.

5. CONCLUSION

We presented a novel nonlinear image reconstruction algorithm which is conceptually similar to filtered back projection, but is not limited by a restriction to linear filters. Preliminary results indicate that the method can reduce reconstruction artifacts. We expect that the method can be improved by making the filters a function of both the cross-section pixel and the projection data.

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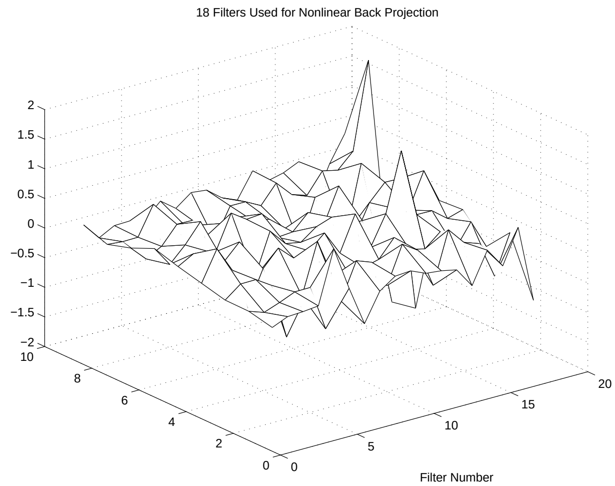

Figure 3: Plot of 18 different filters used in back projection operation. Each filter has 9 taps. Notice that some filters are impulsive while others are smoothing functions.

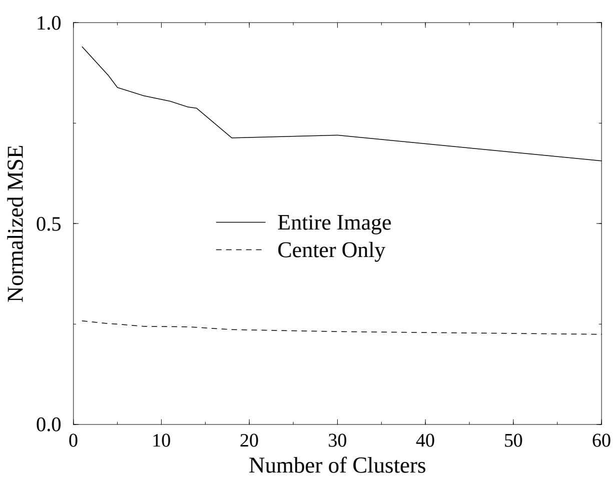

Figure 4: Plot of mean squared reconstruction error versus number of clusters. A value of represents the mean squared error achieved by filtered back projection Notice that the error decreases more rapidly for the entire image since this includes the large discontinuity along the support boundary of the object.

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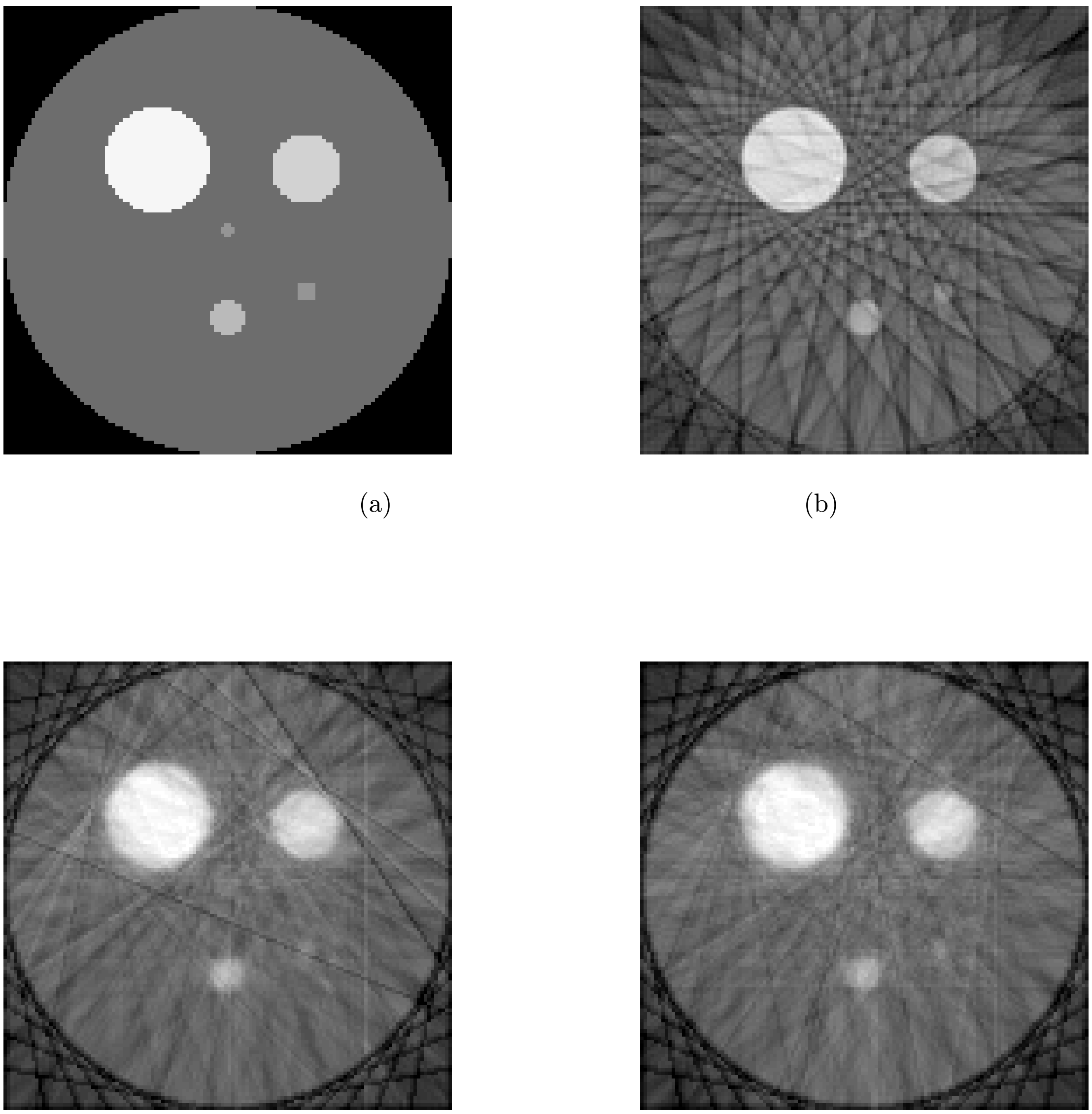

(a) (b)

(c) (d)

Figure 2: (a) Original phantom used for simulations; (b) Filtered back projection reconstruction; (c) Nonlinear back projection result using 18 clusters; (d) Nonlinear back projection result using 60 clusters.