AN OS METHOD FOR DECOMPOSITION OF CYCLIC COMPONENT SIGNALS

Aleksej Makarov

Swiss Federal Institute of Technology (EPFL)
Microprocessor and Interface Laboratory (LAMI)
1015 Lausanne, Switzerland
e-mail: aleksej.makarov@di.epfl.ch

1. INTRODUCTION

A cyclic component signal is a nonstationary signal defined as superposition of a trend, of one or more almost-periodicities and an uncorrelated stochastic component. The cycles of almost-periodicities are not, in contrast to periodicities, the shifted replicas of each other, but vary over time in wavelength, amplitude and shape. The almost-periodic components of a signal are often referred to as cyclic components [1] .

While the classical methods mainly decompose this kind of nonstationary signals into trend, cyclic and stochastic components using the information about the wavelength (the Box & Jenkins SARIMA model, Trend/-Seasonal Dynamic Linear Model [2], filter banks with fixed cut-off frequencies), they fail when this information is unavailable. This is the case when the signal is generated by an unknown physical process, and when the trend cannot be considered as piecewise stationary. If instead of being piecewise stationary, the trend is, say, ramplike, with randomly selected slopes over arbitrary intervals, its spectral content can burry the spectrum of a low-amplitude cyclic component. In such circumstances, the spectral techniques such as the Short Time Fourier Transform fail to provide the information about the wavelength.

Recently, we proposed a new class of methods for determining the average cycle width in cyclic component [3, 4]. These methods are based a multiscale extraction of order statistics (OS), namely of extrema and/or quasiextrema. We applied it by adjusting only two parameters to decomposition of cyclic component signals in various applications such as heart-rate [5] and business statistics records. These parameters are the 'maximum' wavelength T_{max} and the length of the observed block of data N.

For a strictly cyclic component with period T and a large range of trend slopes, it can be shown that the average wavelength T_{AV} converges to T for sufficiently large N. Hence, the method is unbiased. Sparse abrupt changes of the trend do not impair the estimation of T_{AV}. Under constant signal level and arbitrary noise distribution (null hypothesis), the (false alarm) probability of detecting a cyclic or composite signal can be shown to be independent of the functional form and the variance of the noise distribution. Accordingly, the method can be considered nonparametric under null hypothesis (no location shift).

2. DESCRIPTION OF THE METHOD

2.1. Detection of Characteristic Points

A cycle can be represented as a pattern of nonzero and zero slopes. For example, a noiseless sinusoidal cycle contains one positive and one negative slope, as well as zero slopes in between, corresponding to the cycle extrema. Similarly, cyclic component signals of other functional forms contain zero-slope points, which obviously correspond to the possible extremum points. so it is more difficult to find their extrema. One can extract consistent monotonic microtrends within a cycle, and find the time instants where these microtrends change their monotonic behaviour. Such time instants will be referred to as *characteristic points.*

Let us suppose that we use a $M+1-sample$ sliding window for detection of characteristic points, such that $(M+1)T_s = \Delta t$, where T_s is the sampling period and Δt the time duration of the interval covered by the window.

Then, an ascending microtrend, for example, is detected as the following $M+1$-sample pattern:

$$\underbrace{\overbrace{x(k-M)}^{\min\{x\}_{t-\Delta t}} \quad x(k-M+1) \quad \ldots\ldots\ldots \quad x(N)}_{\text{previous window} \quad \{x\}_{t-\Delta t}} \quad \overbrace{x(k)}^{\max\{x\}_t} \tag{1}$$

It can be shown that the pattern (1) is equivalent to the simultaneous increase in both extrema when a $M-$ *sample* sliding window shifts from position $t-1$ to position t, which allows detection of consistent trends in noise.

For the detection of a decreasing microtrend, the extrema in the pattern (1) should be replaced by their opposites.

When running the sliding window over the signal, we form the following binary output $i(t)$: if either an ascending or descending microtrend is detected, $i(t) = 1$, otherwise $i(t) = 0$. The characteristic points are detected at time t as patterns $\{i(t-1), i(t)\} = \{0, 1\}$ or $\{i(t-1), i(t)\} = \{1, 0\}$

2.2. Estimation of the Average Wavelength

Let us consider the case of a cyclic component superimposed on a ramp, as shown in Figure 1. For simplicity, the cyclic component is chosen to be a strictly periodic sinusoid with period T and amplitude a. The ascending ramp is assumed to have a slope b, such that $0 < b \leq \frac{2\pi a}{T}$. The $(M+1)$-sample patterns allow the detection of cycle extrema for windows spreading from points A to B, C to D, and C to E, because B, D and E are the maxima, while A, C and C are the minima of the respective $(M+1)$-sample windows. On the contrary, the windows CF and CG never meet the pattern (1). These windows have lengths equal to an integer multiple of the period T. For such windows, if the current sample $x(k)$ is the maximum of the latest cycle, and hence the overall window maximum, the dropped out sample $x(k-M)$ is the maximum of one of the past cycles, and cannot be the window minimum, so pattern (1) cannot be satisfied and no characteristic point can be detected. Similar behaviour of the characteristic point detector can be derived for descending trends with slope b, such that $0 > b \geq -\frac{2\pi a}{T}$.

By using a range of window sizes from M_{min} to $M_{max} \gg T$, the number of detected characteristic points over an interval of N samples, where $N > M_{max}$, presents a cyclical behaviour, as presented in Figure 2. The wavelength T can be computed as the distance between two neighbouring peaks.

2.3. Decomposition of Composite Signals

As stated in the first section, a bottom-up strategy can be used to evaluate the general trend once the information about the wavelength is available. Under the assumption of small trend changes over a single wavelength, the general trend $\tau(i)$ can be reconstructed as an irregularly sampled sequence, with the

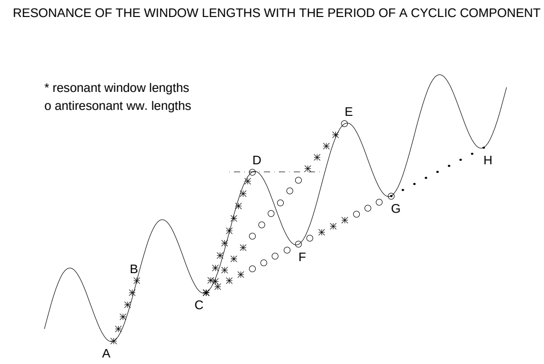

Figure 1: *The characteristic points cannot be detected with the windows whose size equals an odd multiple of the half-period, $M+1 = (2n-1)\frac{T}{2}$, where n is a positive integer. Such windows are referred to as antiresonant. Note that the window lengths which allow the detection of characteristic points are denoted by asterisks, while those which miss the detection are marked with circles.*

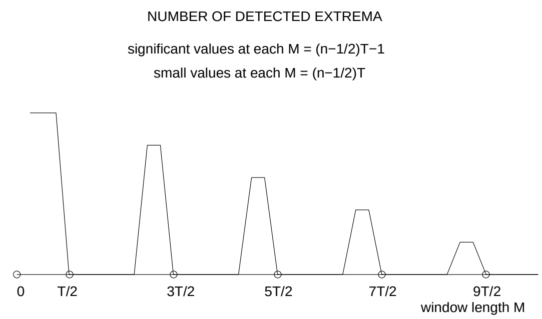

Figure 2: *The number of characteristic points changes cyclically with window length M. The average wavelength, T_{AV}, can be estimated by observing the distance between peaks.*

sampling period corresponding to the estimated wavelength. Assuming that a cycle is symmetrically distributed around zero, the trend value $\tau(K)$ for the observed K^{th} cycle in the data block is equal to the mean value of the composite signal over the wavelength T_K:

$$\tau(K) = \sum_{i=K-\lfloor\frac{T_K}{2}\rfloor}^{K+\lfloor\frac{T_K}{2}\rfloor} x(i) \tag{2}$$

where the cycle index K can be expressed as the lowest integer larger than or equal to the ratio of the observed sample index i and the sum of all previous wavelengths T_J, $J \leq K$

$$K = \lceil \frac{i}{\sum_{J=1}^{K} T_J} \rceil$$

The assumption about symmetrical distribution of the cycle values around zero has been reported in the literature [2] as the zero-sum constraint. As a matter of fact, it implies that the mean value of the cyclic component $\omega(i)$ over an interval $(K - \lfloor \frac{T_K}{2} \rfloor, K + \lfloor \frac{T_K}{2} \rfloor)$ is zero.

The trend values $\tau(i)$ can be interpolated between irregularly sampled points $(K, \tau(K))$ in order to be defined for all signal samples. The easiest way to do it is to use a linear interpolation. The lack of smoothness at the ends of linear segments can be removed by a small moving average filter (typically 3-sample long).

Consequently, the cyclic component $\omega(i)$ can be computed as

$$\omega(i) = x(i) - \tau(i) \tag{3}$$

In the case of several cyclic components, the above procedure can be applied iteratively. For illustration, if two cyclic components $\omega_1(i)$ and $\omega_2(i)$ are present in the composite signal $x(i)$, and if their wavelengths differ enough, the following algorithm is proposed.

1. Segment the signal $x(i)$ into blocks of data of N samples, $N \gg T_{max}$.
2. Estimate the average wavelength of $\omega_1(i)$ for each block of $x(i)$. It is assumed that $\omega_1(i)$ has shorter wavelengths than $\omega_2(i)$.
3. Extract the trend $\tau_1(i)$ by averaging the signal $x(i)$ and interpolating over wavelengths.
4. Extract $\omega_1(i) = x(i) - \tau_1(i)$.
5. Estimate the average wavelength of $\omega_2(i)$ for each block of $\tau_1(i)$.
6. Extract the trend $\tau(i)$ by averaging the signal $\tau_1(i)$ and interpolating over wavelengths.
7. Extract $\omega_2(i) = \tau_1(i) - \tau(i)$.

The extension to additional cyclic components is straightforward.

3. EFFECT OF NOISE

The opposite of a signal which is nonstationary with respect to location signal is a stationary one, i.e., a constant level corrupted with noise. Such a case will be referred to as null hypothesis. Let us find the probability of detection of characteristic points under null hypothesis (the false alarm probability). From pattern (1), if the noise is independent and uncorrelated, the probability that the actual sample $x(t)$ is the window maximum is $\frac{1}{M+1}$. Similarly, the probability that the sample $x(t - \Delta t)$ is the window minimum, under condition that $x(t)$ is the maximum, is $\frac{1}{M}$.

Then the probability of detecting a nonzero slope trend is

$$P_T = \frac{2}{(M+1)M} \tag{4}$$

Hence, the characteristic points are detected in stationary noise with probability

$$P_c = P_T(1 - P_T), \tag{5}$$

independently of the noise distribution. The number of (false) characteristic points in a data block of N samples is therefore

$$N_F = P_c(N - M - 1) \tag{6}$$

If the ascending and descending microtrends are distinguished, the number of false characteristic points is reduced. In such a case, once a characteristic point is detected at time k, no other characteristic point can be detected until time $k + M + 1$. As a matter of fact, the binary sequence $i(t)$ between k and $k + M$ has zero values, due to the trailing extremum $x(k)$, which is no more located on one of the window edges, as in the pattern (1). Hence, the expected number of detected (false) characteristic points on interval of a finite size $N > M$ would be

$$N_F = \frac{P_c(N - M - 1)}{1 + P_c M}. \tag{7}$$

The evolution of N_F with increasing the window length M is shown in Figure 3 with dashed line. A nonstationarity test can be performed by computing the χ^2 statistic with the obtained curve. Thus, the null hypothesis can be rejected or accepted with certain probability.

4. ILLUSTRATIONS

4.1. A synthetic composite signal

In the uppermost diagram of Figure 4, we show the decomposition of a synthetic composite signal, obtained by superposing a sinusoid of constant amplitude $a = 1$ and period $T = 10$ over a trend consisting of a rectangular impulse and the lower half of an ellipse. The signal contains $N = 200$ samples.

In the middle and the lower parts of Figure 4, the extracted components are shown. The dotted line represents the original components. As expected, the largest errors occur around steepest trends.

Note that similar results could have been obtained using polynomial regression, but with a priori knowledge of the order of the polynomial model and of the location of the singular points in the signal. The Bayesian

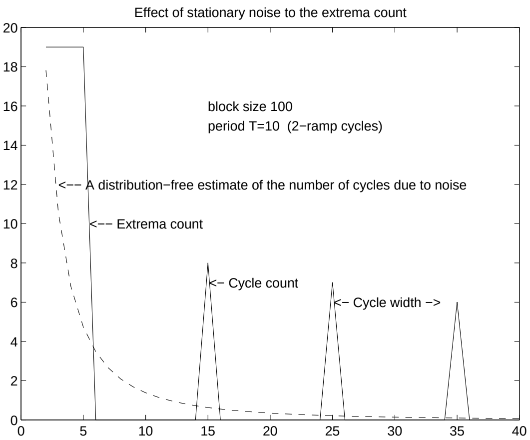

Figure 3: *The number of characteristic points (the ordinate), detected for a range of window lengths M (the abscissa) can be used for a nonstationarity test, which should be satisfied for composite signals.*

algorithms, such as the Dynamic Constant Model [2], can also produce satisfactory results using the strategy described in Introduction, but after adjusting parameters assigned to the trend and the periodicity variances. Note that these parameters do not correspond to the real component variances if the abrupt trend changes are to be distinguished from cyclic changes.

4.2. An economic time series

Economic time series often exhibit a more or less evident cyclic behaviour. The car sales are known to exhibit a complex annual pattern. The most favourable season for car sales is the spring, and to a certain extent, the autumn. The number of sold cars per month in United States, from January 1978 to February 1988 [6] is shown in the uppermost diagram of Figure 5. The signal is scanned with a number of windows in order to detect the characteristic points. The window lengths range from three to sixty-four months. The series is found to be nonstationary with respect to location, using the test described in Section 3. Hence, there are trends with nonzero slopes, and the wavelength estimation described above makes sense. The average cycle length is estimated to be around 12 months, namely 12 ± 4 months within 96% confidence intervals, under assumption of Gaussian distribution of distances between successive peaks of detected characteristic points (drawn in the upper part of Figure 6). This indicates that the car sales follow a varying but consistent yearly seasonal pattern. On the basis of the estimated cycle length, the decomposition procedure described above is performed. In the middle and the lower parts of Figure

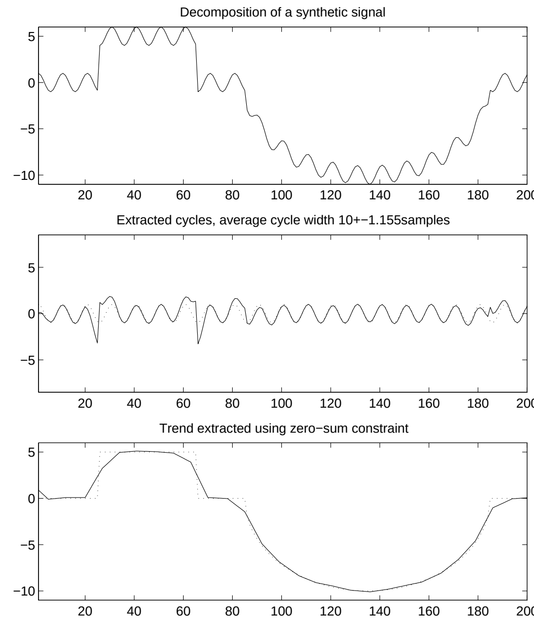

Figure 4: *The original signal (top of the figure), the extracted cyclic and trend components (middle and bottom diagrams). The extracted components are compared with respective original components. Most of the time the extracted components fit the original ones, except around abrupt trend changes.*

5, the extracted components are shown.

For comparison, the Fourier periodogram for the car sales data is shown in Figure 7. The data are smoothed by the Blackman window in order to reduce Gibbs effect. The abscissa in the bottom diagram is expressed as the number of cycles per year. The first two significant lobes occur at rates of approximately one third and two cycles per year, respectively. The annual cycle component cannot be distinguished, mostly because of the spread of the ramplike trend spectral content over the Fourier frequencies. The Fourier periodograms should be mainly applied to detrended signals, necessitating thus an additional technique which is, in the presence of periodicities, a problem per se.

5. REFERENCES

[1] M.West. Bayesian inference in cyclical component dynamic linear models. *Journal of American Statistical Association*, 90(432):1301–1312, December 1995.

[2] M.West and J. Harrison. *Bayesian forecasting and dynamic models.* Springer-Verlag, 1989.

[3] A. Makarov. Discrete frequency tracking in nonstationary signals using joint order statistics technique. In *Proceedings of the IEEE-SP international symposium on*

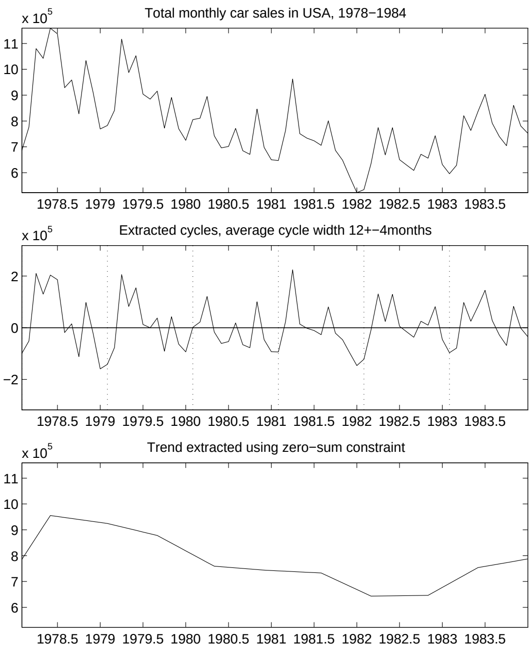

Figure 5: *The original signal (top of the figure), the extracted cyclic and trend components (middle and bottom diagrams).*

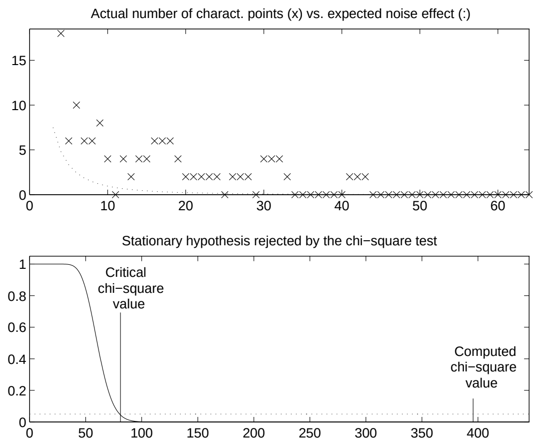

Figure 6: *The number of characteristic points vs. window length (top of the figure) is used as a nonstationarity test. The stationarity with respect to location is rejected as a result the χ^2 test comparing the theoretical and the actual number of characteristic points per month.*

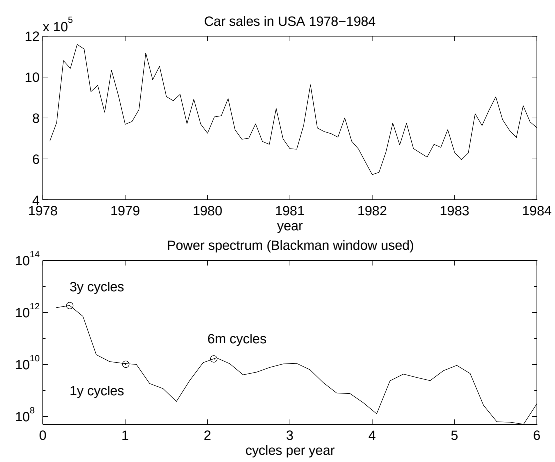

Figure 7: *The Fourier periodogram fails to correctly estimate the frequency (the inverse of the wavelength) of dominant cyclic component in the presence of trends.*

time-frequency and time-scale analysis, pages 441–444, Paris, 1996.

[4] A. Makarov. Periodicity retrieval from nonstationary signals. In *Signal Processing VIII*, pages 1949–1952, Trieste, 1996.

[5] A. Makarov and G. Thonet. Rank order based decomposition and classification of heart rate signals. In *First European Conference on Signal Analysis and Prediction*, Prague, 1997. to be published.

[6] T.E.Dielman. *Applied Regression Analysis for Business and Economics*. The Duxbury Series in Statistics and Decision Sciences. PWS-KENT Publishing Company, Boston, 1991.

[7] C.W.J.Granger and R.Engle. Applications of spectral analysis in economics. In D.R.Brillinger and P.R.Krishnaiah, editors, *Time Series in the Frequency Domain*, volume 3 of *Handbook of Statistics*, chapter 5, pages 93–110. North-Holland, 1983.