NONLINEAR FILTERS AND RAPIDLY INCREASING/DECREASING SIGNALS CORRUPTED WITH NOISE

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ABSTRACT

Here we analyse some specific properties of several well known nonlinear filters applied to processing of ramp edges or rapidly increasing/decreasing linear parts of signals corrupted by Gaussian or mixed noise. It is shown that depending upon the slope, noise variance, filter type and scanning window size the efficiency of noise suppression and spike removal varies in rather wide range and can differ greatly from that one predicted on basis of standard approach to its analysis carried out for constant signals. Quantitative evaluations based on numerical simulation and partly confirmed by analytical derivations are presented.

1. INTRODUCTION

Traditionally nonlinear filters are well known to be efficient in removing impulses and preserving step edges [3,4]. In particular, one of the most known nonlinear filters, the standard median one, is able to reject almost half of spiky values in a sample formed by elements belonging to the current position of a scanning window and it preserves any noncorrupted step edge. But the efficiency of Gaussian noise suppression for standard median filter is essentially worse than for usual mean filter having the same window size. Just this obstacle was the reason for the design of other nonlinear signal processing algorithms combining the advantages of standard median and mean filters. Among the most popular it is possible to mention Hodges-Lehmann, Wilcoxon, α-trimmed and median hybrid filters [2,4].

The results of quantitative estimations of their characteristics - Gaussian noise suppression efficiency and impulsive noise removal ability - are rather widely presented in fundamental books [1,4], but these comparisons are mainly done for constant signals or homogeneous regions of images. One can expect that ramps (i.e., rapidly increasing/decreasing linear parts of signals) should not cause any problems for nonlinear filters and that their behaviour in ramps would be similar to their behaviour in constant signal regions. Interestingly this is not true. These kinds of signals, however, are rather typical for tracking and control systems, for example, in radar applications where the trajectory of the tracked object might be continuously ascending or descending [5]. Such a trajectory would cause the signal under processing to be at least almost linearly increasing or decreasing.

A ramp corrupted with noise changes the performance of the median and other nonlinear filters. In this paper we present the results of empirical studies made with different filters and corrupted ramps. We shall also discuss the reasons for the behaviour of the filters with these corrupted ramps. The discussed filters include mainly nonlinear filters such as median, Wilcoxon, Hodges-Lehmann, α-trimmed and median hybrid ones. The mean filter is used for comparison.

Section 2 deals with analysis of Gaussian noise suppression efficiency of considered 1-D signal processing algorithms and some empirical and analytical results are presented. Then, the case of spike presence in the scanning window is discussed in Section 3 and it is shown that it results in specific undesirable effects in the output signal. Some peculiarities are illustrated and confirmed by examples. Finally, the conclusions follow.

2. ANALYSIS OF GAUSSIAN NOISE SUPPRESSION EFFICIENCY FOR RAMPS

Obviously, the properties of filtering algorithms depend upon several factors: the filter type and its scanning window size, the properties of signals and statistical characteristics of noise. The numerical simulation results and reasonable preliminary assumptions permitted to predict that the characteristics of output signals do not depend directly upon the values of ascending signals first derivative and root mean square (rms) of additive Gaussian noise but on their ratio. So while performing simulations we have not changed the rms of additive Gaussian noise but varied the first derivative of the ascending signal, which can also be expressed as the slope (the angle) of the ramp. The tangent of the angle is the vertical distance of adjacent samples. The original noncorrupted ramps all had nonnegative first derivates. The angle of the ramp was alternated from 0 to 89 degrees. The ramp was first corrupted with additive noise and then filtered. The filtering was repeated sufficient number of times and then the variance and mean of these results were estimated and analysed.

The simulations were done for several scanning window sizes typical for many practical applications. The window size was alternated from 5 to 11. Table 1 shows the output variance of the filtered signals for some of these simulations. Note that one sample for $n = 5, 7$ and two samples for $n = 9, 11$ were trimmed from both ends of the ordered data in the α-trimmed filter.

We found out that although with small angles the differences between the different filters are not that significant, but when the angle is increased the performance of all nonlinear filters gets worse but in different degree.

In the case of the median filter the output variance can be analytically computed using the following Proposition [6].

Proposition 1 *Let the input values X_b, in the window B of a stack filter $S_f(\cdot)$ defined by a positive Boolean function $f(\cdot)$ be independent random variables having the distribution functions $\Phi_b(t)$, respectively. Then the output distribution function $\Psi(t)$ of the stack filter $S_f(\cdot)$ is*

$$\Psi(t) = \sum_{x \in f^{-1}(0)} \prod_{b \in B} (1 - \Phi_b(t))^{x_b} * \Phi_b(t)^{1-x_b}$$

where $f^{-1}(0)$ is the pre-image of 0, i.e., $f^{-1}(0) = \{x | f(x) = 0\}$.

From the output distribution function we have computed the probability density function. Knowing the density function we calculated the output mean and variance. The results obtained analytically confirmed the accuracy of our numerical simulations.

Angle 30° / Filter	Window size 5	7	9	11
mean	1.997	1.433	1.112	0.909
median	2.944	2.211	1.800	1.544
Wilcoxon	2.166	1.574	1.215	0.994
Hodges-Lehmann	2.164	1.544	1.205	0.980
α-trimmed	2.296	1.795	1.322	1.163
median hybrid	2.964	2.519	2.472	2.673

Angle 60° / Filter	Window size 5	7	9	11
mean	1.998	1.430	1.112	0.910
median	3.534	3.072	2.904	2.081
Wilcoxon	2.249	1.679	1.290	1.055
Hodges-Lehmann	2.240	1.647	1.284	1.073
α-trimmed	2.448	2.123	1.531	1.482
median hybrid	4.377	5.381	6.703	7.861

Angle 80° / Filter	Window size 5	7	9	11
mean	1.998	1.429	1.110	0.911
median	7.437	7.436	7.432	7.451
Wilcoxon	2.529	1.857	1.453	1.192
Hodges-Lehmann	2.538	2.016	1.481	1.298
α-trimmed	3.037	3.025	1.894	1.903
median hybrid	9.533	9.940	9.995	10.01

Table 1: The effects of the angle of slope to the filtering efficiency. The variance of the noise is 10.0.

Utilizing Proposition 1 we can obtain the output distribution function for a median of length five which is

$$\begin{aligned}\Psi(x) = \; & 6\Phi_1(x)\Phi_2(x)\Phi_3(x)\Phi_4(x)\Phi_5(x) \\ & -3\textstyle\sum_{i<j<k<l} \Phi_i(x)\Phi_j(x)\Phi_k(x)\Phi_l(x) \\ & +\textstyle\sum_{i<j<k} \Phi_i(x)\Phi_j(x)\Phi_k(x),\end{aligned}$$

where $i, j, k, l \in \{1, 2, 3, 4, 5\}$ and Φ_i, $1 \le i \le 5$, are the cumulative distribution functions of the five samples in the filter window. If the noise is Gaussian distributed with variance σ^2, and the angle of slope is θ then these distribution functions will be

$$\begin{aligned}\Phi_i(x) &= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-1/2\left(\frac{t-(i-3)\tan\theta}{\sigma}\right)^2} dt \\ &= \int_{-\infty}^{x-(i-3)\tan\theta} \frac{1}{\sqrt{2\pi}\sigma} e^{-1/2\left(\frac{t}{\sigma}\right)^2} dt,\end{aligned}$$

for $1 \leq i \leq 5$. The output probability density function is by definition $\frac{d}{dx}\Psi(x)$.

$$\begin{aligned}\frac{d}{dx}\Psi(x) = \;& 6\sum_{i=1}^{5}\phi_i(x)\sum_{j<k<l<m}\Phi_j(x)\Phi_k(x)\Phi_l(x)\Phi_m(x)\\ & -3\sum_{i=1}^{5}\phi_i(x)\sum_{j<k<l}\Phi_j(x)\Phi_k(x)\Phi_l(x)\\ & +\sum_{i=1}^{5}\phi_i(x)\sum_{j<k}\Phi_j(x)\Phi_k(x),\end{aligned}$$

where $j,k,l,m \in \{1,2,3,4,5\}\backslash\{i\}$ and $\phi_i(x), 1 \leq i \leq 5$, are the probability density functions of the distribution functions $\Phi_i(x)$, respectively. For Gaussian distributed noise these density functions are

$$\phi_i(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-1/2\left(\frac{x-(i-3)\tan\theta}{\sigma}\right)^2},$$

for $1 \leq i \leq 5$.

Let us compute the limit of the output density function as the angle of the slope is increased towards 90°. As the angle of the slope approaches 90° the value of $\tan\theta$ increases without bounds. First we define two expressions. If the expectation value of two distributions are μ_1 and μ_2, $\mu_1 \geq \mu_2$ and if the distribution functions of these distributions are $\Phi_1(x)$ and $\Phi_2(x)$, respectively, then $\Phi_1(x)$ is said to lie right of $\Phi_2(x)$ and $\Phi_2(x)$ is said to lie left of $\Phi_1(x)$. Similarly density functions are said to lie right/left of each other. The distance between two distributions is $|\mu_1 - \mu_2|$, i.e the absolute difference between their expectation values. The following Lemmas are proven for the calculation of the output density function.

Lemma 1 *Consider Gaussian distributed data. If we have a product where one density function $\phi(x)$ is multiplied by one or more distribution functions and at least one of the distribution functions lies right of the density function then the limit of the product as the distance d between the density function and the distribution function lying right of it tends to infinity is* 0.

Proof. Without loss of generality we may assume that $\phi(x)$ has zero mean. We have to show that

$$\lim_{d\to\infty}\phi(x)\Phi_1(x)\Phi_2(x)\cdots\Phi_n(x) = 0,$$

where the density function is $\phi(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-1/2\left(\frac{x}{\sigma}\right)^2}$ and at least one of the distribution functions is of the form

$$\Phi_i(x) = \int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}\sigma}e^{-1/2\left(\frac{t-d}{\sigma}\right)^2}\,dt.$$

Consider the case of $x < d/2$. For an arbitrary $\varepsilon > 0$ let $K = \frac{4}{\pi\varepsilon} > 0$ and let $d \geq K$. Now since $x < d/2$ and $K > 0$ we have

$$\begin{aligned}& x-d \leq -d/2\\ \Rightarrow\; & -\tfrac{1}{x-d} \leq \tfrac{1}{d/2} \leq \tfrac{1}{K/2} = \tfrac{\pi\varepsilon}{2}.\end{aligned}$$

The distribution functions $\Phi_i(x)$, $1 \leq i \leq 5$, and the density function $\phi(x)$ are all ≥ 0, the distribution functions are ≤ 1 and the density function is $\leq \frac{1}{\sqrt{2\pi}\sigma}$. Also since

$$\begin{aligned}& e^{x^2/2\sigma^2} \geq x^2/2\sigma^2 \Rightarrow \tfrac{1}{e^{-x^2/2\sigma^2}} \leq 2\sigma^2/x^2\\ \Rightarrow\; & \int_{-\infty}^{x}\tfrac{1}{e^{(t-d)^2/2\sigma^2}}\,dt \leq \int_{-\infty}^{x}2\sigma^2/(t-d)^2\,dt.\end{aligned}$$

Now we can estimate the upper bound of the product.

$$\begin{aligned}& |\phi(x)\Phi_1(x)\Phi_2(x)\cdots\Phi_n(x)|\\ \leq\; & \left|\tfrac{1}{\sqrt{2\pi}\sigma}\tfrac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x}\tfrac{1}{e^{(t-d)^2/2\sigma^2}}\,dt\right|\\ \leq\; & \tfrac{1}{2\pi\sigma^2}\left|\int_{-\infty}^{x}2\sigma^2/(t-d)^2\,dt\right|\\ =\; & \tfrac{2\sigma^2}{2\pi\sigma^2}\left|\int_{-\infty}^{x-d}1/t\,dt\right| = \tfrac{1}{\pi}\left(\tfrac{-1}{x-d}-0\right)\\ =\; & \tfrac{1}{\pi}\tfrac{-1}{x-d} < \tfrac{1}{\pi}\tfrac{\pi\varepsilon}{2}\\ =\; & \tfrac{\varepsilon}{2} < \varepsilon.\end{aligned}$$

Now we have shown that for any $x < d/2$,

$$\begin{aligned}& \forall\varepsilon > 0\;\exists K \geq 0:\\ & d \geq K \Rightarrow |\phi(x)\Phi_1(x)\Phi_2(x)\cdots\Phi_n(x)| < \varepsilon.\end{aligned}$$

Since d tends to infinity we can always find such K that we get $x < d/2$, thus we have shown that for any x, $\lim_{d\to\infty}\phi(x)\Phi_1(x)\Phi_2(x)\cdots\Phi_n(x) = 0$.

Lemma 2 *Consider Gaussian distributed data. If we have a product where one density function is multiplied by one or more distribution functions and all of the distribution functions lie left of the density function then the limit of the product as the pairwise distances $d + c_i$ between the density function and the distribution functions tend to infinity is the density function.*

Proof. We have to show that

$$\lim_{d\to\infty}\phi(x)\Phi_1(x)\Phi_2(x)\cdots\Phi_n(x) = \phi(x),$$

where the density function is $\phi(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-1/2\left(\frac{x}{\sigma}\right)^2}$ and the distributions functions are

$$\Phi_i(x) = \int_{-\infty}^{x+d+c_i}\frac{1}{\sqrt{2\pi}\sigma}e^{-1/2\left(\frac{t}{\sigma}\right)^2}\,dt, c_i \geq 0,$$

for all $1 \leq i \leq n$. Now when $d \to \infty$ also $x + d + c_i \to \infty$ and thus $\lim_{d\to\infty}\int_{-\infty}^{x+d+c_i}\frac{1}{\sqrt{2\pi}\sigma}e^{-1/2\left(\frac{t}{\sigma}\right)^2}\,dt = \lim_{x\to\infty}\Phi(x) = 1$, since $\Phi(x)$ is a distribution function. Now we have

$$\begin{aligned}\lim_{d\to\infty}\tfrac{1}{\sqrt{2\pi}\sigma}\; & e^{-1/2\left(\frac{x}{\sigma}\right)^2}\Phi_1(x)\Phi_2(x)\cdots\Phi_n(x)\\ =\; & \phi(x),\end{aligned}$$

the density function of the distribution.

Now we can compute the output density function of the median with filtering window of length five as the distance between the samples tends to infinity. Note that $\frac{d}{dx}\Psi(x)$ consists only of products where at least one of the distribution functions lies right of the density function or all of the distribution functions lie left of the density function. By utilizing the fact that the limit of a sum is the sum of limits, i.e., $\lim_{x\to c}[f(x)+g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$, we can compute $\lim_{\tan\theta\to\infty} \frac{d}{dx}\Psi(x)$ as a sum of the limits of the products in $\Psi(x)$. Now using Lemmas 1 and 2 and we get

$$\begin{aligned}
& \lim_{\tan\theta\to\infty} \tfrac{d}{dx}\Psi(x) \\
= & \lim_{\tan\theta\to\infty} \left\{ 6\sum_{i=1}^{5}\phi_i(x) \sum_{j<k<l<m} \Phi_j(x)\Phi_k(x)\Phi_l(x)\Phi_m(x) \right. \\
& -3\sum_{i=1}^{5} \phi_i(x) \sum_{j<k<l} \Phi_j(x)\Phi_k(x)\Phi_l(x) \\
& \left. +\sum_{i=1}^{5} \phi_i(x) \sum_{j<k} \Phi_j(x)\Phi_k(x) \right\}, \\
= & \; 6\phi_5(x) - 3(4\phi_5(x) + \phi_4(x)) \\
& +6\phi_5(x) + 3\phi_4(x) + \phi_3(x) \\
= & \; \phi_3(x),
\end{aligned}$$

which is the density function of the middle sample. The expectation value of this distribution is 0 and the variance is σ^2. Thus the median does not change the variance of the noise at all. This tendency can also be seen in our numerical simulations.

The degradation in the performance of the median filter as the angle of the ramp is increased can be explained by the fact that with angle increasing and/or noise variance reduction the sequence of samples tends to a root signal not altered during processing. Some degration of noise suppression efficiency occurs also for Wilcoxon, Hodges-Lehmann and α-trimmed filters as well but it is not too great. For the majority of practically important situation it does not exceed about 20 % for Wilcoxon and Hodges-Lehmann filters and 40 % for α-trimmed filter. As for hybrid median filter it has similar properties as median and even worse.

Usually it is supposed that the increase of the filter scanning window size improves the noise suppression efficiency. For the considered situation it really takes place for Wilcoxon, Hodges-Lehmann and α-trimmed filters. It is also valid for standard median and FIR median hybrid filters when the angle is not that large. Specific effects occur for these filters if the angles are increased. For median filter the results almost do not differ because for any scanning window aperture sizes since the input signals tend to be root ones. For the FIR median hybrid filter the situation is even more striking. With scanning window size increasing the efficiency of noise suppression becomes worse. The special characteristics of this filter are due to the structure of the filter algorithm. The FIR median hybrid filter chooses by definition as it's output the median from the set $\left\{(1/k)\sum_{i=1}^{k} X_i, X_{k+1}, (1/k)\sum_{i=k+2}^{n} X_i\right\}$, where n is the size of the window. Now if the angle is high enough the averages of the samples $X_i, i = 1,\dots,k$, will most of time be smaller than the sample X_{k+1} and this sample will be smaller than the average of $X_i, i = k+2,\dots,n$. When the FIR median hybrid filter computes the median from these the result is generally the sample X_{k+1}, i.e., the filter does not change the signal at all.

3. ANALYSIS OF FILTER OUTPUT PECULIARITIES IN THE PRESENCE OF SPIKE

Now let us also demonstrate that the presence of a spike for ramps also results in specific effects. It is very easy to do for standard median filter with the scanning window size equal to 5. Let the initial (input) sequence be the following: $1, 2, 3, 4, 15, 6, 7, 8, 9, 10$, i.e., it is corrupted by a single spike. Then, the output of standard median filter starting from the third sample is the following: $3, 4, 6, 7, 8, 8$. So it is seen that because of the spike, the fifth, sixth and seventh samples of the output sequence differ form the input ones. This specific effect can be interpreted as a dynamic error or specific jitter. If the scanning window size increases the width of such zone of bias also becomes greater.

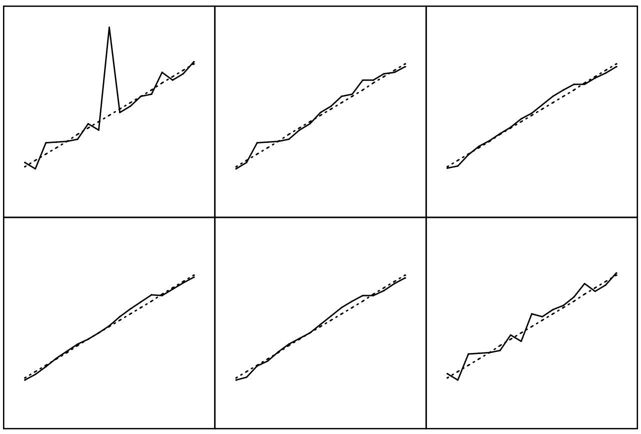

Figure 1: Nonlinear filtering of corrupted ramp signal, $n = 5$.

It is possible to show that similar effects occur for other types of considered nonlinear filters and the biased output areas for them are even wider than for standard median filter. The reason is that their robust properties are worse in comparison to the standard median filter. The place of bias output area depends upon the sign of derivative and the sign of spike. For increasing signal if the spike is negative the bias is also negative and it occurs for samples with index less than that one corrupted by spike.

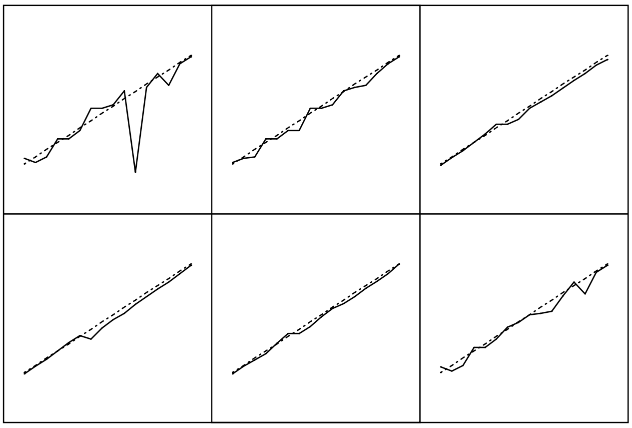

Figure 2: Nonlinear filtering of corrupted ramp signal, $n = 9$.

Figure 1 shows a part of initial ramp corrupted by Gaussian noise and one positive spike and the outputs of considered filters for the scanning window size equal to 5. It is seen that a specific bias is observed for some neighborhood of spiky sample, especially for median hybrid filter output signal. Similarly in Figure 2 the ramp is corrupted with Gaussian noise, but this time there is a negative spike and the length of the filter scanning window is 9.

In Figures 1 and 2 from left to right and top to bottom in the six figures the solid lines are the original corrupted signal and the outputs of median, Wilcoxon, Hodges-Lehmann, α-trimmed, and median hybrid filters, respectively. The dotted line represents the original noncorrupted ramp. It clarifies how effectively each filter has performed. It can be clearly seen that Wilcoxon, Hodges-Lehmann and α-trimmed filters perform quite well in the sense of Gaussian noise suppression although some bias is observed near the spike. On the other hand, the standard median and median hybrid filters are not able to effectively reduce the Gaussian noise from the signal. Comparison of Figures 1 and 2 illustrates the effect of the sign of the spike and the scanning window size to the behaviour of the bias. For a negative spike the bias is located before the spike and for a positive spike after the spike. Also it can be seen that the biased zone increases as the length of filter scanning window is increased from 5 to 9.

The mentioned bias is not desirable. Thus, more effective procedures are needed while processing ramps with impulsive noise. A reasonable solution is to apply adaptive schemes able to detect spikes and to reject them from consideration. The use of nonsymmetrical trimming or selection of other order statistic instead of median also seems to give improvements over the studied "standard" filters.

4. CONCLUSIONS

Nonlinear filters are characterized by specific behavior of output signals for ramps corrupted by additive and impulsive noise, they partly or totally lose their advantages and efficiency of noise suppression. These effects are quantitatively evaluated and demonstrated and they should be taken into account while designing and selecting filters for processing such signals.

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